Exam 3

Name: ______________________________ Section: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. You are allowed to use notes on a single piece of 8.5" x 11" paper, front and back, including formulas and theorems. **You are required to turn this page in with your exam.** You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS). Absolutely no communication device use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ **6** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

2 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ **7** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

3 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ **8** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

4 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ **9** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

5 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ **10** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

1. (5 points) Find the average value of $f(x) = \sin(3x)$ on the interval $[0, \frac{\pi}{2}]$.

   A. $-\frac{1}{3\pi}$

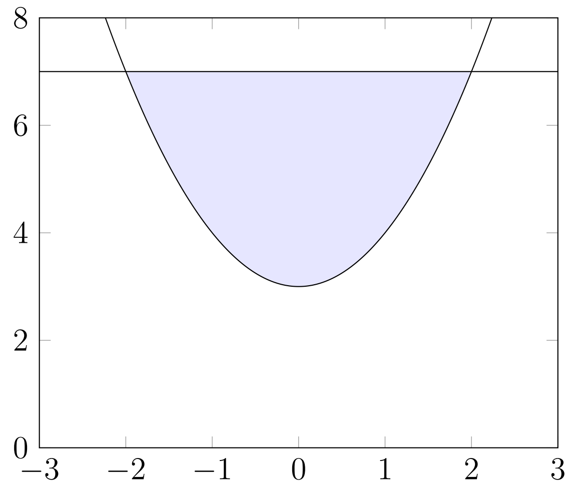
   B. $\frac{1}{3}$

   C. $\frac{2}{3\pi}$

   D. $\frac{1}{6}$

   E. $\frac{3\pi}{2}$

2. (5 points) The region R bounded by $y = x^2 + 3$ and $y = 7$ is shown below.



Consider the solid obtained by rotating R about the **horizontal** line $\mathbf{y = 1}$. Which integral computes the volume of this solid using the **disk/washer method**?

   A. $\pi \int_{-2}^{2} 7^2 - (x^2+3)^2 \, dx$

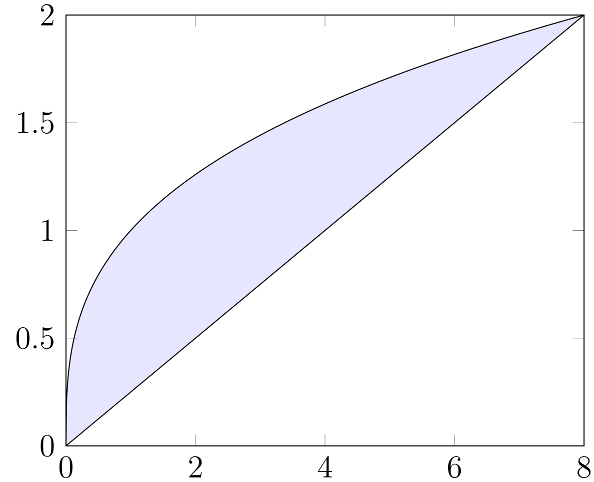
   B. $\pi \int_{-2}^{2} 6^2 - (x^2+2)^2 \, dx$

   C. $\pi \int_{-2}^{2} (x^2-4)^2 \, dx$

   D. $\pi \int_{-2}^{2} (x^2-4)^2 - 1^2 \, dx$

   E. $2\pi \int_{-2}^{2} (x^2+3)(7-1) \, dx$

3. (5 points) The region R bounded by the curves $y = \sqrt[3]{x}$ and $y = \frac{x}{4}$ is shown below.



Consider the solid obtained by rotating R about the $\boldsymbol{x}$**−axis**. Which integral computes the volume of this solid using the **shell** method?

A. $\int_0^2 2\pi y(4y - y^3)\, dy$

B. $\int_0^2 2\pi \left(\frac{y}{4}\right)\left(\sqrt[3]{y} - 8\right)\, dy$

C. $\int_0^2 2\pi((4y)^2 - (y^3)^2)\, dy$

D. $\int_0^2 2\pi y \left(\sqrt[3]{y} - \frac{y}{4}\right)\, dy$

E. $\int_0^2 2\pi \left(\sqrt[3]{y} - \frac{y}{4}\right)^2\, dy$

4. (5 points) Which integral computes the length of the curve $f(x) = \ln(x+3)$ for $0 \le x \le 2$?

A. $\int_0^2 \ln(x+3)\, dx$

B. $\int_0^2 \sqrt{1 + (\ln(x+3))^2}\, dx$

C. $\int_0^2 2\pi \left(\ln(x+3)\right)^2\, dx$

D. $\int_0^2 2\pi \ln(x+3)\sqrt{1 + \left(\frac{1}{x+3}\right)^2}\, dx$

E. $\int_0^2 \sqrt{1 + \left(\frac{1}{x+3}\right)^2}\, dx$

5. (5 points) The curve $y = e^{3x}$ from $x = 1$ to $x = 4$ is rotated about the x-axis. Which integral computes the area of the resulting surface?

   A. $\int_1^4 2\pi e^{3x}\sqrt{1+e^{6x}}\,dx$

   B. $\int_1^4 2\pi e^{3x}\sqrt{1+9e^{6x}}\,dx$

   C. $\int_1^4 2\pi x\sqrt{1+e^{6x}}\,dx$

   D. $\int_1^4 2\pi x\sqrt{1+9e^{6x}}\,dx$

   E. $\int_1^4 \pi e^{6x}\,dx$

6. (5 points) Point masses are placed on the x-axis. m_1 is 5g at $x = -2$, m_2 is 15g at $x = 4$, m_3 is 8g at $x = 6$. Find the center of the mass of the system, correct to two decimal places.

   A. $\bar{x} = 2.67$

   B. $\bar{x} = 4.21$

   C. $\bar{x} = 12.25$

   D. $\bar{x} = 3.50$

   E. $\bar{x} = 32.67$

7. (5 points) Suppose parametric equations for the line segment between $(3,8)$ and $(5,-2)$ have the form $x = a + bt$ and $y = c + dt$. If the curve starts at $(3,8)$ when $t = 0$ and ends at $(5,-2)$ when $t = 1$, find a, b, c, d.

   A. $a = 3, b = 5, c = 8, d = -2$

   B. $a = 8, b = 3, c = -2, d = 5$

   C. $a = 5, b = 3, c = 4, d = -6$

   D. $a = 3, b = 2, c = 8, d = -10$

   E. $a = 5, b = -2, c = 8, d = 10$

8. (5 points) Find the slope of the tangent line to the curve parameterized by $x(t) = 5t + 2$, $y(t) = \sqrt{t + 10}$ at the point $(x, y) = (-3, 3)$.

   A. $\frac{1}{10\sqrt{13}}$

   B. $\frac{1}{10\sqrt{7}}$

   C. -1

   D. $\frac{1}{6}$

   E. $\frac{1}{30}$

9. (5 points) Eliminate the parameter t to find a Cartesian equation satisfied by the curve parameterized by $x(t) = 4 + 7t$, $y(t) = 1 - t^2$.

   A. $4x - 7y^2 = 1$

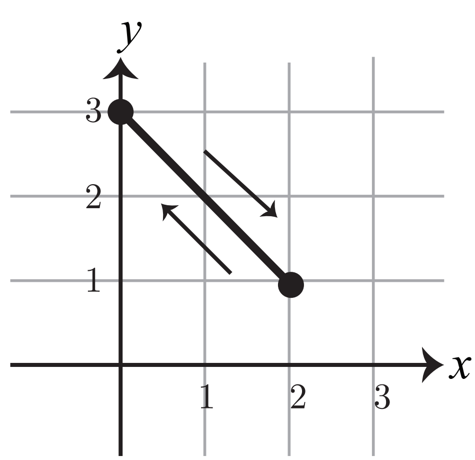
   B. $y = 1 - \left(\frac{7}{x+4}\right)^2$

   C. $y = 1 - \left(\frac{x-4}{7}\right)^2$

   D. $y = 1 - (4 + 7x)^2$

   E. $\left(\frac{x-4}{7}\right)^2 + \left(\frac{y-1}{2}\right)^2 = 1$

10. (5 points) The graph below shows the plot of a parametric curve. Which parameterization is correct?



   A. $x(t) = 2t + 1, \quad y(t) = 3t$

   B. $x(t) = 2\sqrt{t}, \quad y(t) = \ln t, \quad 0 < t \leq 2$

   C. $x(t) = 3t^2 + 1, \quad y(t) = |t|$

   D. $x(t) = \cos^2 t + 1, \quad y(t) = \arctan t$

   E. $x(t) = 1 + \cos t, \quad y(t) = 2 - \cos t$

Free Response Questions

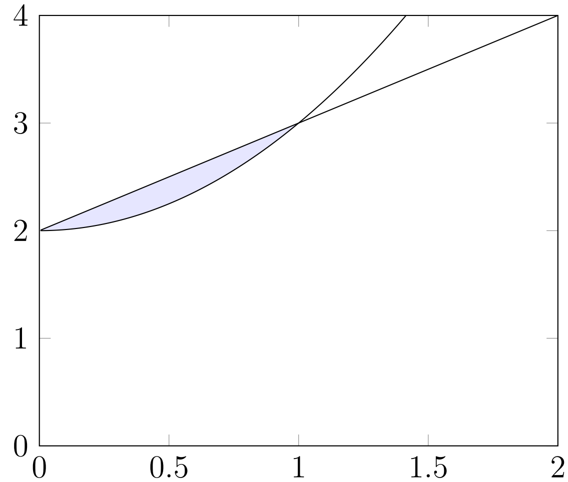
11. The base of a solid is the region enclosed by $y = \sqrt{x}$, $y = 1$, and $x = 16$. Cross-sections perpendicular to the x-axis are squares. Set up an integral which computes the volume of this solid, and then find the volume. Show all steps to compute the integral.

Solution:

The base of each square is $\sqrt{x} - 1$, hence the area of each square is $(\sqrt{x} - 1)^2$. Thus, the volume is calculated as follows

$$V = \int_1^{16} A(x)\ dx = \int_1^{16} (\sqrt{x} - 1)^2\ dx = \int_1^{16} (x - 2\sqrt{x} + 1)\ dx = \frac{1}{2}x^2 - \frac{4}{3}x^{\frac{3}{2}} + x\Big|_1^{16}$$
$$= \frac{351}{6}.$$

12. The region between $y = x^2 + 2$ and $y = x + 2$ is shown below. Let V be obtained by rotating this region about the **vertical** line $x = 7$.



(a) (6 points) Set up but do not evaluate the integral that computes the volume of V using the **disk/washer** method.

Solution:

$$V = \pi \int_2^3 (7 - (y-2))^2 - (7 - \sqrt{y-2})^2 \, dy$$

(b) (4 points) Set up but do not evaluate the integral that computes the volume of V using the **shell** method.

Solution:

$$V = \int_0^1 2\pi(7 - x)\left((x+2) - (x^2+2)\right) \, dx$$

13. Let S be the region between $y = 4x - x^2$ and $y = 2x$. Assume S has uniform density $\rho = 3$.

(a) (8 points) Find the total mass M and the moments M_x and M_y for S. Show all steps clearly. Clearly label each answer.

Solution:

$$m = \rho \int_a^b f(x) - g(x)\ dx = 3\int_0^2 4x - x^2 - 2x\ dx$$
$$= 3\int_0^2 -x^2 + 2x\ dx = -\frac{x^3}{3} + x^2\Big|_0^2 = 4$$

$$M_y = \rho \int_a^b x(f(x) - g(x))\ dx = 3\int_0^2 x(4x - x^2 - 2x)\ dx = 3\int_0^2 -x^3 + 2x^2\ dx$$
$$= 3\left(-\frac{x^4}{4} + \frac{2}{3}x^3\Big|_0^2\right) = 4$$

$$M_x = \frac{\rho}{2}\int_a^b f(x)^2 - g(x)^2\ dx = \frac{3}{2}\int_0^2 (4x - x^2)^2 - (2x)^2\ dx$$
$$= \frac{3}{2}\int_0^2 16x^2 - 8x^3 + x^4 - 4x^2\ dx = \frac{3}{2}\int_0^2 x^4 - 8x^3 + 12x^2\ dx$$
$$= \frac{3}{2}\left(\frac{x^5}{5} - 2x^4 + 4x^3\Big|_0^2\right) = \frac{48}{5}$$

(b) (2 points) Find the center of mass of S.

Solution:

$$\bar{x} = \frac{M_y}{m} = \frac{4}{4} = 1$$
$$\bar{y} = \frac{M_x}{m} = \frac{\frac{48}{5}}{4} = \frac{12}{5}$$

The center of mass is $(\bar{x}, \bar{y}) = (1, \frac{12}{5})$

14. Let C be the curve parameterized by $x(t) = t^3$, $y(t) = t^2$, $0 \leq t \leq 4$.

(a) (6 points) Set up an integral which computes the length of C. Then evaluate your integral to find the length.

Solution:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt = \int_0^4 \sqrt{(3t^2)^2 + (2t)^2}\ dt$$

$$= \int_0^4 \sqrt{9t^4 + 4t^2}\ dt = \int_0^4 t\sqrt{9t^2+4}\ dt$$

Let $u = 9t^2 + 4$, $du = 18t\ dt$ so that our integral becomes

$$\int_4^{148} t\sqrt{u}\frac{1}{18t}\ du = \frac{1}{18} \cdot \frac{2}{3} u^{\frac{3}{2}}\Big|_4^{148} = \frac{1}{27}(148^{\frac{3}{2}} - 4^{\frac{3}{2}}).$$

(b) (4 points) Set up but do **not** evaluate an integral which computes the area of the surface obtained by revolving C about the **x-axis**

Solution:

$$SA = \int_a^b 2\pi r L = \int_0^4 2\pi t^2 \sqrt{(3t^2)^2 + (2t)^2}\ dt$$

15. Let C be the curve parameterized by $x(t) = e^t,\ y(t) = (t-1)^2$.

(a) (5 points) Find the slope of the tangent line to C at the point $(x, y) = (1, 1)$.

Solution:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2(t-1)}{e^t}.$$

Setting $x(t) = e^t = 1$, and $y(t) = (t-1)^2 = 1$, we find that point $(x, y) = (1, 1)$ corresponds to $t = 0$. Therefore,

$$m = \left.\frac{dy}{dx}\right|_{t=0} = \frac{2(0-1)}{e^0} = -2.$$

(b) (5 points) Find the second derivative, $\frac{d^2y}{dx^2}$, in terms of t. (You do not need to simplify your answer).

Solution:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{e^t(2)-2(t-1)e^t}{(e^t)^2}}{e^t} = \frac{4-2t}{e^{2t}}$$