MA123, Chapter 10: Formulas for integrals: integrals, antiderivatives, and the Fundamental Theorem of Calculus (pp. 207-233, Gootman)

`Chapter Goals:`

- Understand the statement of the Fundamental Theorem of Calculus.
- Learn how to compute the antiderivative of some basic functions.
- Learn how to use the substitution method to compute the antiderivative of more complex functions.
- Learn how to solve area and distance traveled problems by means of antiderivatives.

`Assignments:` `Assignment 22` `Assignment 23` `Assignment 24`

So far we have learned about the idea of the integral, and what is meant by computing the definite integral of a function $f(x)$ over the interval $[a, b]$. As in the case of derivatives, we now study procedures for computing the definite integral of a function $f(x)$ over the interval $[a, b]$ that are easier than computing limits of Riemann sums. As with derivatives, however, the definition is important because it is only through the definition that we can understand why the integral gives the answers to particular problems.

▶ Idea of the Fundamental Theorem of Calculus:

The easiest procedure for computing definite integrals is not by computing a limit of a Riemann sum, but by relating integrals to (anti)derivatives. This relationship is so important in Calculus that the theorem that describes the relationships is called the Fundamental Theorem of Calculus.

▶ Computing some antiderivatives:

In previous chapters we were given a function $f(x)$ and we found the derivative $f'(x)$. In this section, we will do the reverse. We will be given a function $f(x)$ that is the derivative of another function $F(x)$ and will compute $F(x)$. In other words find a function $F(x)$ such that $F'(x) = f(x)$. $F(x)$ is called an **antiderivative** of $f(x)$. For example $(x^3)' = 3x^2$ so an antiderivative of $f(x) = 3x^2$ is $F(x) = x^3$.

Note that $F(x) = x^3 + 2$ is also an antiderivative of $f(x) = 3x^2$ because $(x^3 + 2)' = 3x^2$. In general, if $F(x)$ is an antiderivative of $f(x)$, then so is $F(x) + C$ where C is any constant. This leads to the following notation.

Definition of the indefinite integral:

The *indefinite integral* of $f(x)$, denoted by

$$\int f(x)\, dx$$

without limits of integration, is the *general* antiderivative of $f(x)$.

For example, it is easy to check that $\displaystyle\int 3t^2\, dt = t^3 + c,$ where c is any constant.

Recall that the power rule for derivatives gives us $(x^n)' = nx^{n-1}$. We multiply by n and subtract 1 from the exponent. Since antiderivatives are the reverse of derivatives, to compute an the antiderivative we first increase the power by 1, then divide by the new power.

The formulas below can be verified by differentiating the righthand side of each expression.

Some basic indefinite integrals:

1. $\displaystyle\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \qquad n \neq -1$ **2.** $\displaystyle\int e^x\,dx = e^x + C$

$n = -1$ in formula **1** leads to division by zero, but for this special case we may use $(\ln(x))' = \frac{1}{x}$:

3. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$

Rules for indefinite integrals:

A. $\displaystyle\int c\,f(x)\,dx = c\int f(x)\,dx$ **B.** $\displaystyle\int (f(x) \pm g(x))\,dx = \left(\int f(x)\,dx\right) \pm \left(\int g(x)\,dx\right)$

Example 1: Evaluate the indefinite integral $\displaystyle\int (t^3 + 3t^2 + 4t + 9)\,dt$. power rule

$$= \frac{t^4}{4} + \frac{3t^3}{3} + \frac{4t^2}{2} + 9t + C$$

$$= \frac{t^4}{4} + t^3 + 2t^2 + 9t + C$$

Example 2: Evaluate the indefinite integral $\displaystyle\int \frac{6}{\sqrt{t}}\,dt$. $= \displaystyle\int 6t^{-1/2}\,dt$

$$= \frac{6t^{1/2}}{(1/2)} + C = 12t^{1/2} + C$$

Warning: We do not have simple derivative rules for products and quotients, so we should not expect simple integral rules for products and quotients.

Example 3: Evaluate the indefinite integral $\displaystyle\int t^3(t+2)\,dt$. $= \displaystyle\int (t^4 + 2t^3)\,dt$

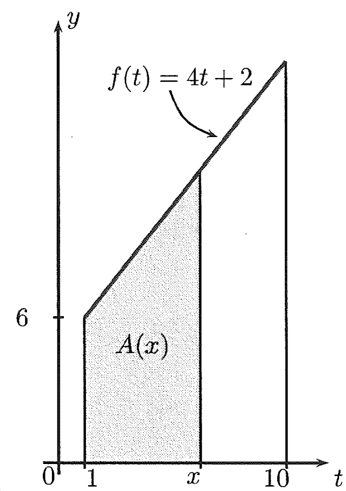
$$= \frac{t^5}{5} + \frac{2t^4}{4} + C = \frac{t^5}{5} + \frac{t^4}{2} + C$$

$\boxed{\textbf{Example 4:}}$ Evaluate the indefinite integral $\displaystyle\int \frac{x^2+9}{x^2}\,dx.$ $= \int 1 + 9x^{-2}$

$$= x + \frac{9x^{-1}}{-1} + C = x - \frac{9}{x} + C$$

We now have some experience computing antiderivatives. We will now see how antiderivatives give us an elegant method for finding areas under curves.

$\boxed{\textbf{Example 5:}}$ Find a formula for $A(x) = \displaystyle\int_1^x (4t+2)\,dt,$ that is, evaluate the definite integral of the function $f(t) = 4t+2$ over the interval $[1,x]$ inside $[1,10]$. (**Hint:** think of this definite integral as an area.) Find the values $A(5), A(10), A(1)$. What is the derivative of $A(x)$ with respect to x?



$$\int_1^x (4t+2)\,dt = \text{Area under } y = 4t+2 \text{ from } t=1 \text{ to } t=x$$

$$= \text{Area of trapezoid}$$

$$= \left(\frac{(4x+2)+(4\cdot 1+2)}{2}\right)(x-1)$$

$$= (2x+4)(x-1) = 2x^2+2x-4 = A(x)$$

$A(5) = 56$

$A(10) = 216$

$A(1) = 0$

$\boxed{\textbf{Observations:}}$ There are two important things to notice about the function $A(x)$ analyzed in Example 1:

$$A(1) = \int_1^1 (4t+2)\,dt = 0 \qquad A'(x) = \frac{d}{dx}\Bigg(\underbrace{\int_1^x (4t+2)\,dt}_{A(x)}\Bigg) = 4x+2.$$

Notice what the last equality says: The instantaneous rate of change of the area under the curve $y = 4t+2$ at $t = x$ is simply equal to the value of the curve evaluated at $t = x$.

Why? $A(x)$ measures the area of some geometric figure. As x increases, the width of the figure increases, and so the area increases. $A'(x)$ measures the rate of increase of the figure. Now, as x increases, the right wall of the figure sweeps out additional area, so the rate at which the area increases should be equal to the height of the right wall.

The following pages will make this idea more precise.

Idea: Suppose that for *any* function $f(t)$ it were true that the area function $A(x) = \int_a^x f(t)\,dt$ satisfies

$$A(a) = \int_a^a f(t)\,dt = 0 \qquad A'(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x).$$

Moreover, suppose that $F(x)$ is any andtiderivative of $f(x)$ (i.e., $F'(x) = f(x) = A'(x)$.) By **The Constant Function Theorem** (Chapter 6), there exists a constant value c such that $F(x) = A(x) + c$.

All these facts put together help us easily evaluate $\int_a^b f(t)\,dt$. Indeed,

$$\begin{aligned}\underline{\int_a^b f(t)\,dt} &= A(b) = A(b) - 0 \\ &= \underline{A(b) - A(a)} = [A(b) + c] - [A(a) + c] \\ &= \underline{F(b) - F(a)}\end{aligned}$$

The above 'speculations' are actually true for any continuous function on the interval over which we are integrating. These results are stated in the following theorem, which is divided into two parts:

The Fundamental Theorem of Calculus:

PART I: Let $f(t)$ be a continuous function on the interval $[a, b]$. Then the function $A(x)$, defined by the formula

$$A(x) = \int_a^x f(t)\,dt$$

for all x in the interval $[a, b]$, is an antiderivative of $f(x)$, that is

$$A'(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

for all x in the interval $[a, b]$.

PART II: Let $F(x)$ be *any* antiderivative of $f(x)$ on $[a, b]$, so that

$$F'(x) = f(x)$$

for all x in the interval $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Special notations: The above theorem tells us that evaluating a definite integral is a two-step process: find *any* antiderivative $F(x)$ of the function $f(x)$ and then compute the difference $F(b) - F(a)$. A notation has been devised to separate the two steps of this process: $F(x)\Big|_a^b$ stands for the difference $F(b) - F(a)$. Thus

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

▶ **Recall some properties of definite integrals:**

1. $\displaystyle\int_a^a f(x)\,dx = 0$

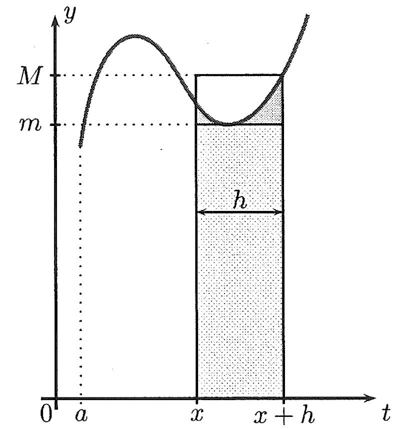
2. $\displaystyle\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

3. $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \left(\int_a^b f(x)\,dx\right) \pm \left(\int_a^b g(x)\,dx\right)$

4. $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

5. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

▶ **An idea of the proof of the Fundamental Theorem of Calculus:**



We already gave an explanation of why the second part of the Fundamental Theorem of Calculus follows from the first one. To prove the first part we need to use the definition of the derivative. More precisely, we must show that

$$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = f(x).$$

For convenience, let us assume that f is a positive valued function. Given that $A(x)$ is defined by $\displaystyle\int_a^x f(t)\,dt$, the numerator of the above difference quotient is

$$A(x+h) - A(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt.$$

Using properties **4.** and **5.** of definite integrals, the above difference equals $\displaystyle\int_x^{x+h} f(t)\,dt$. As the function f is continuous over the interval $[x, x+h]$, the Extreme Value Theorem says that there are values c_1 and c_2 in $[x, x+h]$ where f attains the minimum and maximum values, say m and M, respectively. Thus $m \le f(t) \le M$ on $[x, x+h]$. As the length of the interval $[x, x+h]$ is h, by property **6.** of definite integrals we have that

$$f(c_1)h = mh \le \int_x^{x+h} f(t)\,dt \le Mh = f(c_2)h \qquad \text{or, equivalently,} \qquad f(c_1) \le \frac{\int_x^{x+h} f(t)\,dt}{h} \le f(c_2).$$

Finally, as f is continuous we have that $\lim_{h\to 0} f(c_1) = f(x) = \lim_{h\to 0} f(c_2)$. This concludes the proof.

▶ **Examples illustrating the First Fundamental Theorem of Calculus:**

Example 6: Compute the derivative of $F(x)$ if $\displaystyle F(x) = \int_2^x (t^4 + t^3 + t + 9)\,dt$.

$$\frac{d}{dx}\left(\int_2^x (t^4 + t^3 + t + 9)\,dt\right) = x^4 + x^3 + x + 9$$

Example 7: Compute the derivative of $g(s)$ if $g(s) = \int_5^s \frac{8}{\sqrt{u^2+u+2}}\,du.$

$$g'(s) = \frac{8}{\sqrt{s^2+s+2}}$$

Example 8: Use the chain rule to compute the derivative of $F(x)$ if $F(x) = \int_0^{x^2} 2t\,dt.$

$$F'(x) = 2(x^2)(2x) = 4x^3$$

plug in x^2

derivative of x^2

Example 9: Compute the derivative of $F(x)$ if $F(x) = \int_1^{5x+9} (t^2+7t+3)\,dt.$

$$F'(x) = \left((5x+9)^2 + 7(5x+9)+3\right)\cdot(5)$$

plug in $5x+9$, multiply by derivative of $5x+9$

Example 10: Suppose $f(x) = \int_1^x \sqrt{t^2-7t+12.25}\,dt.$ For which positive value of x does $f'(x)$ equal 0?

$$f'(x) = \sqrt{x^2-7x+12.55} = 0$$

$$\Rightarrow \quad x^2-7x+12.55=0$$

$$\Rightarrow \quad (x-3.5)^2=0$$

$$\Rightarrow \quad x=3.5$$

Example 11: Find the value of x at which $F(x) = \int_3^x (t^8 + t^6 + t^4 + t^2 + 1)\, dt$ takes its minimum on the interval $[3, 100]$. The value of x that gives a minimum of $F(x)$ is ______.

$F'(x) = x^8 + x^6 + x^4 + x^2 + 1$ which is not zero on $[3, 100]$

$\Rightarrow$ no critical values. In fact, $F'(x) > 0$

$\Rightarrow$ $F(x)$ is increasing $\Rightarrow$ Max occurs at right endpoint ... $x = 100$

Example 12: Find the value of x at which $G(x) = \int_{-5}^x (|t| + 2)\, dt$ takes its maximum on the interval $[-5, 100]$. The value of x that gives a maximum of $G(x)$ is ______.

$G'(x) = |x| + 2 > 0 \Rightarrow G$ is increasing

$\Rightarrow$ G attains its max at right endpoint

$x = 100$

▶ **Examples illustrating the Second Fundamental Theorem of Calculus:**

Example 13: Evaluate the integral $\int_0^5 (t^2 + 1) dt$.

$$\int_0^5 (t^2+1)dt = \frac{t^3}{3} + t \Big|_0^5 = \frac{5^3}{3} + 5 - \left(\frac{0^3}{3} + 0\right)$$

$$= \frac{140}{3}$$

Example 14: Evaluate the integral $\int_{-7}^x \left(\frac{1}{t}\right)^2 dt$.

$$\int_{-7}^x \left(\frac{1}{t}\right)^2 dt = \int_{-7}^x t^{-2} dt = -t^{-1}\Big|_{-7}^x = -x^{-1} - 7^{-1}$$

$$= -\frac{1}{x} - \frac{1}{7}$$

Example 15: Evaluate the integral $\int_0^T e^x\,dx$. $= e^x\Big|_0^T = e^T - e^0$

$= e^T - 1$

Example 16: Evaluate the integral $\int_{-2}^{2} (t^5 + t^4 + t^3 + t^2 + t + 1)\,dt$.

$$= \frac{t^6}{6} + \frac{t^5}{5} + \frac{t^4}{4} + \frac{t^3}{3} + \frac{t^2}{2} + t\Big|_{-2}^{2}$$

$$= \left(\frac{2^6}{6} + \frac{2^5}{5} + \frac{2^4}{4} + \frac{2^3}{3} + \frac{2^2}{2} + 2\right) - \left(\frac{(-2)^6}{6} + \frac{(-2)^5}{5} + \frac{(-2)^4}{4} + \frac{(-2)^3}{3} + \frac{(-2)^2}{2} + (-2)\right)$$

$$= \frac{332}{15}$$

Example 17: Evaluate the integral $\int_{-6}^{x} |t|\,dt$.

If $x<0$ then $\int_{-6}^{x} |t|\,dt = \int_{-6}^{x} -t\,dt = -\frac{t^2}{2}\Big|_{-6}^{x} = -\frac{x^2}{2} + \frac{(-6)^2}{2}$

$= -\frac{x^2}{2} + 18$

If $x>0$ then $\int_{-6}^{x} |t|\,dt = \int_{-6}^{0} -t\,dt + \int_{0}^{x} t\,dt = -\frac{t^2}{2}\Big|_{-6}^{0} + \frac{t^2}{2}\Big|_{0}^{x}$

$= 0 + 18 + \frac{x^2}{2} + 0$

$= \boxed{\frac{x^2}{2} + 18}$

Example 18: Evaluate the integral $\int_2^T \left(3u^5 + \frac{7}{u}\right)du$.

$$\int_2^T \left(3u^5 + \frac{7}{u}\right)du = 3\frac{u^6}{6} + \ln|u|\Big|_2^T$$

$$= \frac{T^6}{2} + \ln T - \left(\frac{1}{2}\cdot 2^6 + \ln(2)\right)$$

Example 19: Suppose

$$f(x) = \begin{cases} 2x, & x \leq 2; \\ \dfrac{8}{x}, & x > 2. \end{cases}$$

Evaluate the integral $\displaystyle\int_0^5 f(x)\,dx$.

$$= \int_0^2 2x\,dx + \int_2^5 \frac{8}{x}\,dx = x^2 \Big|_0^2 + 8\ln(x)\Big|_2^5$$

$$= 2^2 - 0^2 + 8\ln(5) - 8\ln(2)$$

$$= 4 + 8\ln(5) - 8\ln(2)$$

▶ **The substitution rule for integrals:** If $u = g(t)$ is a differentiable function whose range is a subinterval I and f is continuous on I, then $\displaystyle\int f(g(t))\,g'(t)\,dt = \int f(u)\,du$.

In case of definite integrals the substitution rule becomes $\displaystyle\int_a^b f(g(t))\,g'(t)\,dt = \int_{g(a)}^{g(b)} f(u)\,du$.

Example 20: Evaluate the integral $\displaystyle\int_0^x (t+9)^2\,dt$.

Let $\boxed{u = t+9 \quad \frac{du}{dt} = 1 \quad du = dt}$

$t = 0 \Rightarrow u = 9$
$t = x \Rightarrow u = x+9$

$$= \int_9^{x+9} u^2\,du = \frac{u^3}{3}\Big|_9^{x+9}$$

$$= \frac{(x+9)^3}{3} - \frac{9^3}{3}$$

$$= \frac{(x+9)^3}{3} - 243$$

Example 21: Evaluate the integral $\displaystyle\int_0^x \sqrt{3t+7}\,dt$.

Let $\boxed{u = 3t+7 \quad \frac{du}{dt} = 3 \quad \frac{du}{3} = dt}$

$t = 0 \Rightarrow u = 7$
$t = x \Rightarrow u = 3x+7$

$$\frac{1}{3}\int_7^{3x+7} u^{1/2}\,du$$

$$= \frac{1}{3}\frac{u^{3/2}}{(3/2)}\Big|_7^{3x+7} = \boxed{\frac{2}{9}(3x+7)^{3/2} - \frac{2}{9}\cdot 7^{3/2}}$$

Example 22: Evaluate the integral $\int_0^x \frac{1}{(5t+4)^2}\,dt.$ $\longrightarrow$ $\int_4^{5x+4} u^{-2}\frac{du}{5} = \frac{-u^{-1}}{5}\Big|_4^{5x+4}$

Let $u = 5t+4$ $\quad t=0 \Rightarrow u=4$

$\frac{du}{dt} = 5$ $\quad t=x \Rightarrow u=5x+4$

$\frac{du}{5} = dt$

$$= -\frac{1}{5u}\Big|_4^{5x+4}$$

$$= -\frac{1}{5(5x+4)} + \frac{1}{5\cdot 4}$$

Example 23: Evaluate the integral $\int_0^5 \sqrt{2t+1}\,dt.$

$u = 2t+1$ $\quad t=0 \Rightarrow u=1$

$\frac{du}{dt} = 2$ $\quad t=5 \Rightarrow u=11$

$\frac{du}{2} = dt$

$$= \int_1^{11} u^{1/2}\frac{du}{2} = \frac{u^{3/2}}{2\cdot(3/2)}\Big|_1^{11}$$

$$= \frac{1}{3}(11)^{3/2} - \frac{1}{3}(1)^{3/2}$$

Example 24: Evaluate the integral $\int_0^1 5\,e^{5x+1}\,dx$

$u = 5x+1$ $\quad x=0 \Rightarrow u=1$

$\frac{du}{dx} = 5$ $\quad x=1 \Rightarrow u=6$

$\frac{du}{5} = dx$

$$= \int_1^6 5e^u\frac{du}{5}$$

$$= e^u\Big|_1^6 = e^6 - e^1$$

Example 25: Evaluate the integral $\int_0^T \frac{2x}{x^2+1}\,dx.$

$$= \int_1^{T^2+1} \frac{\cancel{2x}}{u}\cdot\frac{du}{\cancel{2x}}$$

$u = x^2+1$ $\quad x=0 \Rightarrow u=1$

$\frac{du}{dx} = 2x$ $\quad x=T \Rightarrow u=T^2+1$

$\frac{du}{2x} = dx$

$$= \int_1^{T^2+1} \frac{1}{u}\,du = \ln|u|\Big|_1^{T^2+1}$$

$$= \ln(T^2+1) - \cancel{\ln(1)}^{\,0}$$

$$= \ln(T^2+1)$$

▶ $\boxed{\text{Word problems:}}$

Suppose $v(t)$ measures the velocity of an object at time t. Recall from Chapter 8 that:

(a) $\int_a^b v(t)\,dt$ measures the displacement of the object from $t = a$ to $t = b$. The displacement is the difference between the object's ending point and starting point.

(b) $\int_a^b |v(t)|\,dt$ measures the total distance traveled between $t = a$ and $t = b$.

If $v(t)$ is always positive, *displacement* and *distance traveled* are the same.

Example 26: A train travels along a track and its velocity (in miles per hour) is given by $v(t) = 76t$ for the first half hour of travel. Its velocity is constant and equal to $v(t) = 76/2$ after the first half hour. Here time t is measured in hours. How far (in miles) does the train travel in the second hour of travel?

Distance in 2nd hour $= \int_1^2 |v(t)|\,dt = \int_1^2 \frac{76}{2}\,dt = \int_1^2 38\,dt$

$= 38t\Big|_1^2 = 38(2) - 38(1)$

$= 38$ miles

Example 27: A train travels along a track and its velocity (in miles per hour) is given by $v(t) = 76t$ for the first half hour of travel. Its velocity is constant and equal to $v(t) = 76/2$ after the first half hour. Here time t is measured in hours. How far (in miles) does the train travel in the first hour of travel?

Distance in first hour $= \int_0^1 |v(t)|\,dt = \int_0^{1/2} 76t\,dt + \int_{1/2}^1 38\,dt$

$= 38t^2\Big|_0^{1/2} + 38t\Big|_{1/2}^1$

$= 38(1/2)^2 - 38(0)^2 + 38(1) - 38(1/2) = 28.5$

Example 28: A rock is dropped from a height of 21 feet. Its velocity in feet per second at time t after it is dropped is given by $v(t) = -32t$ where time t is measured in seconds. How far is the rock from the ground one second after it is dropped?

$\int_0^1 |v(t)|\,dt = \int_0^1 32t = 16t^2\Big|_0^1 = 16$

Example 29: Suppose an object is thrown down from a cliff with an initial speed of 5 feet per sec, and its speed in ft/sec after t seconds is given by $s(t) = 10t + 5$. If the object lands after 7 seconds, how high (in ft) is the cliff? **(Hint:** how far did the object travel?)

$$\text{Distance traveled} = \int_0^7 (10t+5)\,dt$$
$$= 5t^2 + 5t \Big|_0^7$$
$$= 5\cdot 7^2 + 5(7) - 5\cdot 0^2 - 5\cdot 0$$
$$= 280 \text{ ft}$$

Example 30: A car is traveling due east. Its velocity (in miles per hour) at time t hours is given by

$$v(t) = -2.5t^2 + 10t + 50.$$

How far did the car travel during the first five hours of the trip?

$$\text{Distance traveled} = \int_0^5 (-2.5t^2 + 10t + 50)\,dt$$
$$= \frac{-2.5}{3}t^3 + \frac{10t^2}{2} + 50t \Big|_0^5$$
$$= -\frac{2.5}{3}(5)^3 + 5(5)^2 + 50(5) - (0+0+0)$$
$$= 270.8\overline{33}$$