

MA123, Chapter 2: Change, and the idea of the derivative (pp. 17-45, Gootman)

Chapter Goals:

- Understand average rates of change.
- Understand the ideas leading to instantaneous rates of change.
- Understand the connection between instantaneous rates of change and the derivative.
- Know the definition of the derivative at a point.
- Use the definition of the derivative to calculate derivatives.
- Understand the connection between a position function, a velocity function, and the derivative.
- Understand the connection between the derivative and the slope of a tangent line.

Assignments:

Assignment 02 Assignment 03

Roughly speaking, Calculus describes how quantities change, and uses this description of change to give us extra information about the quantities themselves.

► **Average rates of change:** We are all familiar with the concept of velocity (speed): If you drive a distance of 120 miles in two hours, then your average velocity, or rate of travel, is $120/2 = 60$ miles per hour. In other words, the average velocity is equal to the ratio of the distance traveled over the time elapsed:

$$\text{average velocity} = \frac{\text{distance traveled}}{\text{time elapsed}} = \frac{\Delta s}{\Delta t}$$

In general, the quantity $\frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$ is called the **average rate of change of y with respect to x** .

Note: Often, a change in a quantity q is expressed by the symbol Δq (you should not think of this as Δ times q , but rather as one quantity!).

Note: Finding average rates of change is important in many contexts. For instance, we may be interested in knowing how quickly the air temperature is dropping as a storm approaches, or how fast revenues are increasing from the sale of a new product.

Note: In this course we use the terms “speed” and “velocity” for the same concept. This is not the case in some other courses. Thus “instantaneous speed” and “instantaneous velocity” have the same meaning, and “average speed” and “average velocity” have the same meaning.

Example 1: A train travels from city A to city B . It leaves A at 10:00 am and arrives at B at 2:30 pm. The distance between the cities is 150 miles. What was the average velocity of the train in miles per hour (mph)?

Do you think the train was always traveling at the same speed?

$$\begin{aligned} \text{Average Velocity} &= \frac{\text{Distance traveled}}{\text{Time Elapsed}} = \frac{150 \text{ miles}}{4.5 \text{ hours}} \\ &= \frac{100}{3} \text{ miles/hour} = 33.\overline{33} \text{ mph} \end{aligned}$$

Example 2: A train leaves station A at 8:00 am and arrives at station B at 10:00 am. The train stops at station B for 1 hour and then continues to station C. It arrives at station C at 3:00 pm. The average velocity from A to B was 40 mph and the average velocity from B to C was 50 mph. What was the average velocity from A to C (including stopping time)?

$$\text{Total time elapsed} = 7 \text{ hours} \quad (8:00 \text{ am to } 3:00 \text{ pm})$$

$$\text{Distance A to B} = (40 \text{ mph})(2 \text{ hours}) = 80 \text{ miles}$$

$$\text{Distance B to C} = (50 \text{ mph})(4 \text{ hours}) = 200 \text{ miles}$$

$$\text{Average Velocity A to C} = \frac{280}{7} = 40 \text{ mph}$$

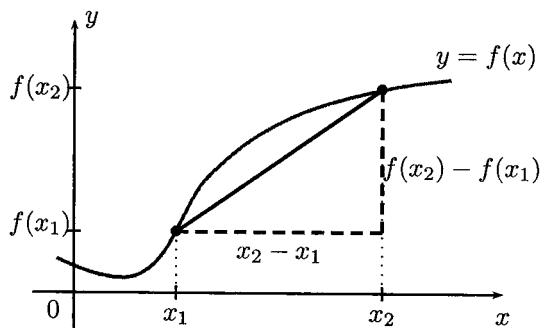
Generally, in computing average rates of change of a quantity y with respect to a quantity x , there is a function that shows how the values of x and y are related.

► **Average rates of change of a function:**

The **average rate of change** of the function $y = f(x)$ between $x = x_1$ and $x = x_2$ is

$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

The average rate of change is the slope of the **secant line** between $x = x_1$ and $x = x_2$ on the graph of f , that is, the line that passes through $(x_1, f(x_1))$ and $(x_2, f(x_2))$.



Example 3: Find the average rate of change of $g(x) = 2 + 4(x - 1)$ with respect to x as x changes from -2 to 5 . Could you have predicted your answer using your knowledge of linear equations?

Since $g(x)$ is linear, $g(x)$ is identical to any of its secant lines

$$\text{So } \text{AROC} = \text{slope of } g(x) = 4$$

Alternatively:

$$g(-2) = 2 + 4(-2 - 1) = -10$$

$$g(5) = 2 + 4(5 - 1) = 18$$

$$\text{So } \text{AROC} = \frac{g(5) - g(-2)}{5 - (-2)} = \frac{18 - (-10)}{5 - (-2)} = \frac{28}{7} = 4$$

Example 4: Find the average rate of change of $k(t) = \sqrt{3t + 1}$ with respect to t as t changes from 1 to 5 .

$$\text{AROC} = \frac{k(5) - k(1)}{5 - 1} \quad \text{now } k(1) = \sqrt{3 \cdot 1 + 1} = \sqrt{4} = 2$$

$$k(5) = \sqrt{3 \cdot 5 + 1} = \sqrt{16} = 4$$

$$\text{So } \text{AROC} = \frac{4 - 2}{5 - 1} = \frac{2}{4} = \frac{1}{2}$$

Example 5: A particle is traveling along a straight line. Its position at time t seconds is given by $s(t) = 2t^2 + 3$. Find the average velocity of the particle as t changes from 0 seconds to 4 seconds.

Displacement = $\Delta s = s(4) - s(0)$
 $s(4) = 2 \cdot 4^2 + 3 = 35$
 $s(0) = 2 \cdot 0^2 + 3 = 3$
 so $\Delta s = 35 - 3 = 32$

Average Velocity
 $= \frac{\Delta s}{\Delta t} = \frac{32}{4} = 8$

Example 6: Let $g(x) = \frac{1}{x}$. Find a value for x such that the average rate of change of $g(x)$ from 1 to x equals $-\frac{1}{10}$.

AROC from 1 to x is $\frac{g(x) - g(1)}{x - 1}$

Now,

$g(x) - g(1) = \frac{1}{x} - \frac{1}{1} = \frac{1-x}{x}$

so $AROC = \frac{\left(\frac{1-x}{x}\right)}{x-1} = \frac{1-x}{x(x-1)}$

We're Given $AROC = -\frac{1}{10}$

so

$\frac{1-x}{x(x-1)} = -\frac{1}{10} \Rightarrow$

$\xrightarrow{\text{cross multiply}} -10(1-x) = x(x-1)$

$10(x-1) = x(x-1)$

$10 = x$

Example 7: Find the average rate of change of $k(t) = t^3 - 5$ with respect to t as t changes from 1 to $1+h$.

$AROC = \frac{k(1+h) - k(1)}{1+h - 1} = \frac{k(1+h) - k(1)}{h}$

Now, $k(1) = 1^3 - 5 = 1 - 5 = -4$

and $k(1+h) = (1+h)^3 - 5 = -4 + 3h + 3h^2 + h^3$

so $AROC = \frac{(-4 + 3h + 3h^2 + h^3) - (-4)}{h}$

$= \frac{3h + 3h^2 + h^3}{h} = \frac{(3 + 3h + h^2)h}{h} = 3 + 3h + h^2$

→ why? Foil + Foil again:

$(1+h)^3 = (1+h)(1+h)(1+h) = (1+2h+h^2)(1+h)$

$= (1+2h+h^2) + (1+2h+h^2)h$

$= 1 + 2h + h^2 + h + 2h^2 + h^3$

$= 1 + 3h + 3h^2 + h^3$

► **Instantaneous rates of change:** The phrase 'instantaneous rate of change' seems like an oxymoron, a contradiction in terms like the phrases 'thunderous silence' or 'sweet sorrow'. However, because of your experience with traveling and looking at speedometers, both the concept of *average velocity* and the concept of *velocity at an instant* have an intuitive meaning to you. The connection between the two concepts is that if you compute the average velocity over smaller and smaller time periods you should get numbers that are closer and closer to the speedometer reading at the instant you look at it.

Definition: The instantaneous rate of change is defined to be the result of computing the average rate of change over smaller and smaller intervals.

The following algebraic approach makes this idea more precise.

Algebraic approach: Let $s(t)$ denote, for sake of simplicity, the position of an object at time t . Our goal is to find the instantaneous velocity at a fixed time $t = a$, say $v(a)$. Let the first value be $t_1 = a$, and the second time value $t_2 = a + h$. The corresponding positions of the object are

$$s_1 = s(t_1) = s(a) \quad s_2 = s(t_2) = s(a + h),$$

respectively. Thus the average velocity between times $t_1 = a$ and $t_2 = a + h$ is

$$\frac{s_2 - s_1}{t_2 - t_1} = \frac{s(a + h) - s(a)}{h}.$$

To see what happens to this average velocity over smaller and smaller time intervals we let h get closer and closer to 0. This latter process is called *finding a limit*. Symbolically:

$$v(a) = \lim_{h \rightarrow 0} \frac{s(a + h) - s(a)}{h}.$$

Note: We can discuss the instantaneous rate of change of any function using the method above. When we discuss the instantaneous rate of change of the position of an object, then we call this change the instantaneous velocity of the object (or the velocity at an instant). We often shorten this phrase and speak simply of the velocity of the object. Thus, the velocity of an object is obtained by computing the average velocity of the object over smaller and smaller time intervals.

Example 8: A particle is traveling along a straight line. Its position at time t is given by $s(t) = 5t^2 + 3$.

Find the velocity of the particle when $t = 4$ seconds.

Average Velocity

$$\frac{s(4+h) - s(4)}{h} = \frac{(5(4+h)^2 + 3) - (5 \cdot 4^2 + 3)}{h}$$

$$= \frac{5(16 + 8h + h^2) + 3 - 5 \cdot 16 - 3}{h}$$

$$= \frac{5 \cdot 8h + 5h^2}{h} = \frac{(40 + 5h)h}{h} = 40 + 5h$$

Velocity:

Let $h \rightarrow 0$.

$$40 + 5h \rightarrow 40 + 5 \cdot 0 = 40$$

Example 9: A particle is traveling along a straight line. Its position at time t is given by $s(t) = 5t^2 + 3$.

Find the velocity of the particle when $t = 2$ seconds.

Average Velocity

$$\frac{s(2+h) - s(2)}{h} = \frac{(5(2+h)^2 + 3) - (5 \cdot 2^2 + 3)}{h}$$

$$= \frac{5(4 + 4h + h^2) + 3 - 5 \cdot 4 - 3}{h}$$

$$= \frac{20h + 5h^2}{h} = \frac{(20 + 5h)h}{h} = 20 + 5h$$

Velocity

Let $h \rightarrow 0$

$$20 + 5h \rightarrow 20 + 5 \cdot 0 = 20$$

Notice using $v(t) = 10t$, we can recover the answers in #8 & 9 by taking $v(4) = 10 \cdot 4 = 40$ & $v(2) = 10 \cdot 2 = 20$

The approach we will see now has the tremendous advantage that it yields a formula for the instantaneous velocity of this object as a function of time t .

Example 10: A particle is traveling along a straight line. Its position at time t is given by $s(t) = 5t^2 + 3$.

Find the velocity of the particle as a function of t .

Average Velocity

$$\frac{s(t+h) - s(t)}{h} = \frac{(5(t+h)^2 + 3) - (5t^2 + 3)}{h}$$

$$= \frac{5(t^2 + 2th + h^2) + 3 - 5t^2 - 3}{h}$$

$$= \frac{10th + 5h^2}{h} = \frac{(10t + 5h)h}{h} = 10t + 5h$$

Velocity:

$$v(t) = \lim_{h \rightarrow 0} \frac{\Delta s}{\Delta t}$$

$$= \lim_{h \rightarrow 0} 10t + 5h = 10t + 5 \cdot 0$$

$$= 10t = v(t)$$

Note: Even if you have a formula for a quantity, knowing how the quantity is changing can give you extra information that is not obvious from the formula. For example, if $s(t)$ denotes the position of a ball being thrown up into the air, how high does the ball go? Observe that when the ball is going up it has positive velocity, because its height is increasing in time, whereas when the ball is going down it has negative velocity, because its height is decreasing in time. Thus, the instant at which the ball reaches its highest point is exactly the one when its velocity is 0.

Example 11:

A particle is traveling along a straight line. Its position at time t is given by $s(t) = -2t^2 + 6t + 5$.

- Find the velocity of the particle as a function of t ;
- When is the velocity of the particle equal to 5 feet per second?
- When is the velocity of the particle equal to 0 feet per second?

a)

$$\text{Average Velocity} = \frac{s(t+h) - s(t)}{h}$$

Now,

$$s(t+h) = -2(t+h)^2 + 6(t+h) + 5$$

$$= -2t^2 - 4th - 2h^2 + 6t + 6h + 5$$

So

$$\frac{s(t+h) - s(t)}{h} = \frac{(-2t^2 - 4th - 2h^2 + 6t + 6h + 5) - (-2t^2 + 6t + 5)}{h}$$

$$= \frac{-4th - 2h^2 + 6h}{h} = \frac{(-4t - 2h + 6)h}{h}$$

$$= -4t - 2h + 6$$

So Velocity is obtained letting $h \rightarrow 0$:

$$v(t) = -4t + 6$$

(b) Want $v(t) = 5$.
Velocity is given & time is unknown.

So $-4t + 6 = 5$
 $\Rightarrow -4t = -1 \Rightarrow t = \frac{1}{4} \text{ sec.}$

(c) Want $v(t) = 0$

So $-4t + 6 = 0$
 $4t = 6$
 $t = \frac{6}{4} = \frac{3}{2} \text{ secs}$

→ Notice the answer to #12 can be recovered from #13 by substituting $k=3$: Inst. ROC = $4 \cdot 3 + 1 = 13$

Example 12: Find the instantaneous rate of change of $g(k) = 2k^2 + k - 1$ at $k=3$.

A ROC:
$$\frac{g(3+h) - g(3)}{h} = \frac{(2(3+h)^2 + (3+h) - 1) - (2 \cdot 3^2 + 3 - 1)}{h}$$

$$= \frac{2(9 + 6h + h^2) + 3 + h - 1 - 2 \cdot 9 - 3 + 1}{h} = \frac{12h + 2h^2 + h}{h} = \frac{13h + 2h^2}{h}$$

$$= \frac{(13 + 2h)h}{h} = 13 + 2h \Rightarrow \text{IROC: Let } h \rightarrow 0:$$

$$13 + 2 \cdot 0 = \boxed{13}$$

Example 13: Find the instantaneous rate of change of $g(k) = 2k^2 + k - 1$ as a function of k .

A ROC:
$$\frac{g(k+h) - g(k)}{h} = \frac{[2(k+h)^2 + (k+h) - 1] - [2k^2 + k - 1]}{h}$$

$$= \frac{2(k^2 + 2kh + h^2) + k + h - 1 - 2k^2 - k + 1}{h} = \frac{4kh + 2h^2 + h}{h} = \frac{(4k + 2h + 1)h}{h}$$

$$= 4k + 2h + 1$$

Inst. ROC: Let $h \rightarrow 0$:
$$\boxed{4k + 1 = \text{Inst. ROC.}}$$

► **The derivative:** The derivative of f with respect to x , at $x = x_1$, is the instantaneous rate of change of f with respect to x , at $x = x_1$, and is thus given by the formula

$$\lim_{h \rightarrow 0} \frac{f(x_1 + h) - f(x_1)}{h}$$

Now just drop the subscript '1' from the x in the above formula, and you obtain the instantaneous rate of change of f with respect to x at a general point x . This is called the *derivative of f at x* and is denoted with $f'(x)$:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Alternative notations:

As we remarked earlier, a change in a quantity q is often expressed by the symbol Δq (you should not think of this as Δ times q , but rather as one quantity!). Thus the above formulas are often rewritten as

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x} = \frac{df}{dx}$$

Given a general function $f(x)$, it is often common to think in terms of $y = f(x)$ so that the above formulas are often rewritten as

$$y' = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

A preview:

Often, the information you have about a quantity is not about the quantity itself, but about its rate of change. This means that you know the derivative of a function, and want to find the function. This occurs frequently in Physics. Newton's formula gives information about the *acceleration* of an object, that is, about the rate of change of velocity with respect to time. From this, one can often get information about the velocity, the rate of change of position with respect to time, and then information about the position itself of the object.

Example 14: Let $f(x) = mx + b$ be an arbitrary linear function (here m and b are constants).

Use the definition of the derivative to show that $f'(x) = m$.

Algebraic Method

$$\text{AROC} = \frac{f(x+h) - f(x)}{h} = \frac{(m(x+h) + b) - (mx + b)}{h}$$

$$= \frac{mx + mh + b - mx - b}{h} = \frac{mh}{h} = m$$

Now let $h \rightarrow 0$, But AROC doesn't depend on h , so

$$f'(x) = \lim_{h \rightarrow 0} m = m$$

Geometric: AROC = Slope

since $f(x)$ is linear.

So AROC = m ,

Then

$$f'(x) = \lim_{h \rightarrow 0} m = m.$$

Example 15: Let $f(x) = ax^2 + bx + c$ be an arbitrary quadratic function (here a , b and c are constants).

Use the definition of the derivative to show that $f'(x) = 2ax + b$.

$$\text{AROC: } \frac{f(x+h) - f(x)}{h} = \frac{(a(x+h)^2 + b(x+h) + c) - (ax^2 + bx + c)}{h}$$

$$= \frac{ax^2 + 2axh + ah^2 + bx + bh + c - ax^2 - bx - c}{h}$$

$$= \frac{2axh + ah^2 + bh}{h} = \frac{(2ax + ah + b)h}{h}$$

$$= 2ax + ah + b$$

$$\text{So } f'(x) = \lim_{h \rightarrow 0} 2ax + ah + b = 2ax + a \cdot 0 + b.$$

$$= \boxed{2ax + b = f'(x)}$$

Example 16: Let $g(x) = 2x^2 + x - 1$. Find a value c between 1 and 4 such that the average rate of change of $g(x)$ from $x = 1$ to $x = 4$ is equal to the instantaneous rate of change of $g(x)$ at $x = c$.

AROC From 1 to 4 is

$$\frac{g(4) - g(1)}{4 - 1} = \frac{(2 \cdot 4^2 + 4 - 1) - (2 \cdot 1^2 + 1 - 1)}{3}$$

$$= \frac{33}{3} = 11$$

IROC: Use result from #15.

$$g'(x) = 2 \cdot 2x + 1 = 4x + 1$$

$$\text{So } g'(c) = 4c + 1$$

Want c where AROC = IROC

$$\text{So } 11 = 4c + 1 \Rightarrow 10 = 4c \Rightarrow c = \frac{10}{4} = \frac{5}{2}$$

[We were asked to find such a c between 1 and 4.
we double check that our c does lie between 1 & 4.]

Example 17: Let $g(x) = 2x^2 + x - 1$. Find a value x_0 such that $g'(x_0) = 4$.

Write $g(x) = 2x^2 + x - 1 = ax^2 + bx + c$,
 So $a = 2, b = 1, c = -1$.
 Use #15 to find $g'(x)$.

$g'(x) = 2ax + b = 2(2)x + 1 = 4x + 1$

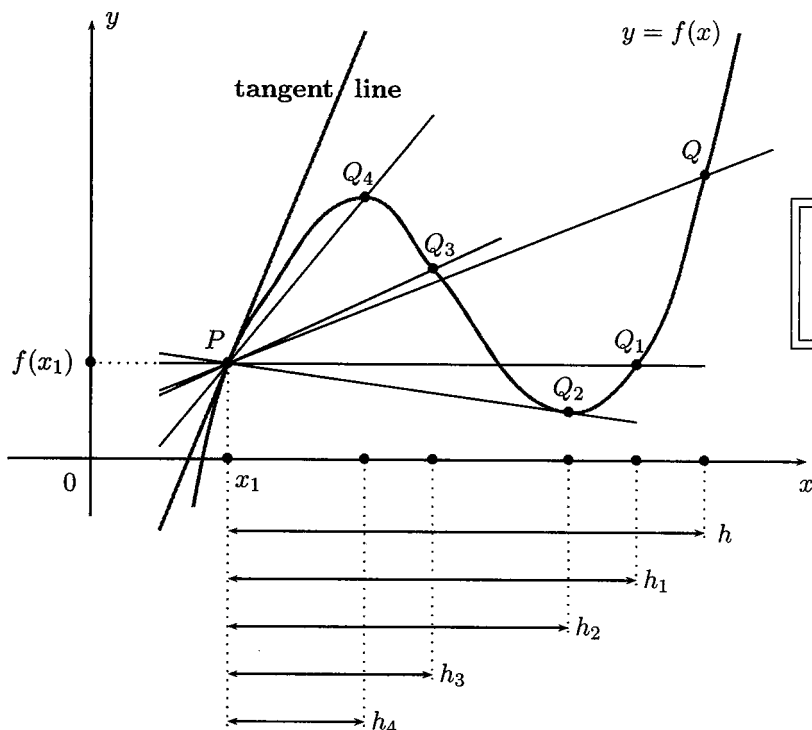
want x_0 where $g'(x_0) = 4$.
 So $4x_0 + 1 = 4$
 $\Rightarrow 4x_0 = 3$
 $x_0 = \frac{3}{4}$

► **The geometric meaning of derivatives:**

Let $f(x)$ be a function and consider its graph $y = f(x)$ in the Cartesian coordinate system. Consider the following two points $P(x_1, f(x_1))$ and $Q(x_1 + h, f(x_1 + h))$ on the graph of f . It should be clear from our discussion that the slope of the straight (secant) line through P and Q

$$\frac{f(x_1 + h) - f(x_1)}{(x_1 + h) - x_1} = \frac{f(x_1 + h) - f(x_1)}{h} = \frac{\Delta y}{\Delta x}$$

is nothing but the average rate of change of f with respect to x , as the variable changes from x_1 to $x_1 + h$. As the value of h changes, you get a succession of different lines, all passing through $P(x_1, f(x_1))$. As h gets closer and closer to 0, the lines get closer and closer to what you probably intuitively think of as the *tangent line* to the graph of f at the point $P(x_1, f(x_1))$.



Thus the derivative of f at x_1 , namely $f'(x_1)$, is the slope of the tangent line to the graph of f at the point $P(x_1, f(x_1))$.

Definition: The **tangent line** to the graph of a function f at a point $P(x_1, f(x_1))$ on the graph is the line passing through the point $P(x_1, f(x_1))$ and with slope equal to $f'(x_1)$. Thus, the **equation of the tangent line at the point $P(x_1, f(x_1))$** is:

$$y = f(x_1) + f'(x_1)(x - x_1)$$

Example 18: Let $f(x) = x^2 + x + 14$.

What is the value of x for which the slope of the tangent line to the graph of $y = f(x)$ is equal to 5?

$$\begin{array}{l|l}
 f(x) = ax^2 + bx + c & f'(x) = \text{slope of tangent line.} \\
 \Rightarrow a=1, b=1, c=14, & \text{Want } x \text{ where} \\
 \text{so } f'(x) = 2ax + b & f'(x) = 5 \\
 = 2x + 1 & \text{so } 2x + 1 = 5 \\
 & \Rightarrow 2x = 4 \\
 & \boxed{x = 2}
 \end{array}$$

Example 19: Let $f(x) = x^2 + x + 14$.

What is the value of x for which the tangent line to the graph of $y = f(x)$ is parallel to the x -axis?

Tangent Line parallel to x -axis.
 \Rightarrow Tangent Line Horizontal.
 \Rightarrow Slope = 0.
 Now, $f'(x) = 2x + 1$
 so $2x + 1 = 0$
 $2x = -1 \Rightarrow x = -\frac{1}{2}$

Notice For $f(x) = ax^2 + bx + c$
 vertex occurs where
 tangent line is horizontal.

ie, where $f'(x) = 2ax + b = 0$
 $\Rightarrow x = -b/2a$

This is one of many algebraic formulas that can be recovered quickly if you know Calculus.

Linear approximation of a function at a point:

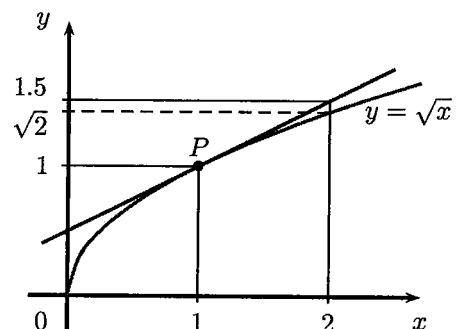
The importance of computing the equation of the tangent line to the graph of a function f at a point $P(x_1, f(x_1))$ lies in the fact that if we look at a portion of the graph of f near the point P , it becomes indistinguishable from the tangent line at P . In other words, the values of the function are close to those of the linear function whose graph is the tangent line. For this reason, the linear function whose graph is the tangent line to $y = f(x)$ at the point $P(x_1, f(x_1))$ is called the **linear approximation** of f near $x = x_1$. We write

$$f(x) \approx f(x_1) + f'(x)(x - x_1) \quad \text{near } x = x_1.$$

For example, to find an approximate value for $\sqrt{2}$ we can proceed as follows. Consider the function $f(x) = \sqrt{x}$. We will learn in Chapter 4 that the derivative of f is given by $f'(x) = \frac{1}{2\sqrt{x}}$. Thus the equation of the tangent line at the point $P(1, 1)$ can easily be seen to be

$$y = 1 + \frac{1}{2}(x - 1) \quad \text{so that } \sqrt{x} \approx 1 + \frac{1}{2}(x - 1) \quad \text{near } x = 1.$$

If we substitute $x = 2$ into the above approximation we obtain the value $3/2 = 1.5$, which is very close to $\sqrt{2} = 1.4142\dots$ The picture on the right illustrates the situation.



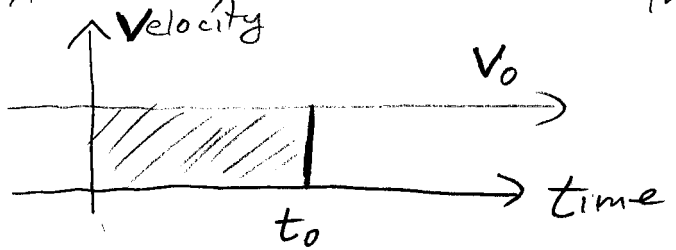
Side notes

$$\text{From average velocity} = \frac{\text{distance traveled}}{\text{time Elapsed}},$$

we find

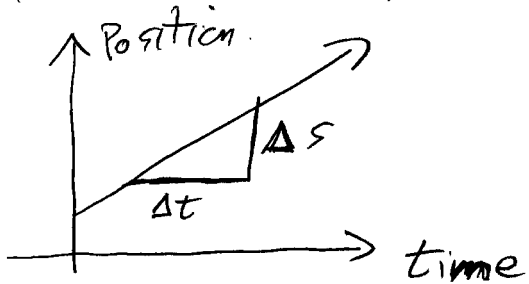
$$\text{Distance traveled} = (\text{Average Velocity}) \cdot (\text{time Elapsed})$$

If velocity is constant and we plot velocity against time, then distance traveled can be thought of as area under the velocity curve.



This interpretation will be very important in later chapters.

Likewise, if the position of the object is a linear function of time, $s(t) = a \cdot t + b$, then the average velocity can be interpreted as the slope of the position function.



These two interpretations hint at the Fundamental Theorem of Calculus, which describes a fundamental relationship between slopes of curves and areas bounded by curves.