MA123, Chapter 5: Formulas for Derivatives (pp. 83-102, Gootman)

`Chapter Goals:`	• Know and be able to apply the formulas for derivatives. • Understand the *chain rule* and be able to apply it. • Know how to compute higher derivatives.
`Assignments:`	`Assignment 08` `Assignment 09`

In this chapter we learn general formulas to compute derivatives without resorting to the definition of the derivative each time. Of course the validity of these formulas is based upon the definition of the derivative, along with facts about limits.

We encourage you to memorize the procedure for finding the derivative in terms of words.

We will also give each differentiation formula using both the 'prime' notation and the $\frac{d}{dx}$ (or Leibniz) notation.

► **Derivative of a constant function:**

- If $f(x) = c$, a constant, then $f'(x) = 0$.
- $\frac{d}{dx}(c) = 0$.
- The derivative of a constant is zero.

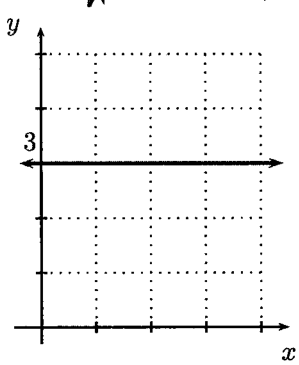
Why? Many reasons.
(a) $f'(x) =$ slope of tangent line, but $f(x) =$ constant $\Rightarrow$ Graph is horizontal line.
(b) $f'(x)$ measures change in $f(x)$, but $f(x) =$ constant $\Rightarrow$ $f(x)$ doesn't change
(c) Definition: $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0$

Example 1:
Let $f(x) = 3$. Find $f'(x)$.

$f(x) =$ constant, so $f'(x) = 0$

i.e, $\frac{d}{dx}(3) = 0$

i.e. $(3)' = 0$.



► **Power rule:**

- If $f(x) = x^n$, then $f'(x) = n\,x^{n-1}$.
- $\frac{d}{dx}(x^n) = n\,x^{n-1}$.
- To take the derivative of x raised to a power, you multiply in front by the exponent and subtract 1 from the exponent.

Note: The remarkable fact about the power rule is that it works not only for exponents that are natural numbers, but for any exponents. The quotient rule (see later) lets us prove the power rule for negative exponents.

When you compute the derivative of a function containing $\sqrt[n]{}$, always change the $\sqrt[n]{}$ to a fractional exponent, so that you can take advantage of the power rule (most likely in combination with the chain rule!). For instance, $f(x) = \sqrt[4]{x-3} = (x-3)^{1/4}$.

Example 2: Find the derivative of each of the following functions with respect to the appropriate variable:

(a) $y = x^4$ $\quad y' = (x^4)' = 4x^{4-1} = 4x^3$

(b) $g(s) = s^{-2}$ $\quad g'(s) = (s^{-2})' = (-2)s^{-2-1} = -2s^{-3}$

(c) $h(t) = t^{3/4}$ $\quad h'(t) = (t^{3/4})' = \left(\frac{3}{4}\right)t^{\frac{3}{4}-1} = \frac{3}{4}t^{-1/4}$

Rewrite each function as a power of x!

Example 3: Find the derivative of each of the following functions with respect to x:

(a) $y = \frac{1}{x^5} = x^{-5}$, so $y' = (x^{-5})' = -5x^{-5-1} = -5x^{-6}$

(b) $g(x) = \sqrt[3]{x} = x^{1/3}$, so $g'(x) = (x^{1/3})' = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-2/3}$

(c) $h(x) = \frac{1}{\sqrt[5]{x}} = x^{-1/5}$, so $h'(x) = (x^{-1/5})' = -\frac{1}{5}x^{-\frac{1}{5}-1}$

$= -\frac{1}{5}x^{-6/5}$

▶ **The constant multiple rule:** Let c be a constant and $f(x)$ be a differentiable function.

- $(cf(x))' = c(f'(x))$.
- $\frac{d}{dx}\Big(cf(x)\Big) = c\frac{d}{dx}\Big(f(x)\Big)$.
- The derivative of a constant times a function equals the constant times the derivative of the function. In other words, when computing derivatives, multiplicative constants can be pulled out of the expression.

Example 4: Find the derivative of each of the following functions with respect to x:

(a) $f(x) = 2x^3$, $\quad f'(x) = (2x^3)' = 2\cdot(x^3)' = 2\cdot 3x^{3-1}$

$= 6x^2$

(b) $h(x) = \frac{1}{3x^2} = \frac{1}{3}\cdot\frac{1}{x^2} = \frac{1}{3}x^{-2}$, so

$h'(x) = \left(\left(\frac{1}{3}\right)x^{-2}\right)' = \left(\frac{1}{3}\right)(x^{-2})' = \left(\frac{1}{3}\right)(-2x^{-2-1})$

$= -\frac{2}{3}x^{-3}$

▶ $\boxed{\textbf{The sum rule:}}$ Let $f(x)$ and $g(x)$ be differentiable functions.

- $(f(x)+g(x))' = f'(x)+g'(x)$.
- $\frac{d}{dx}\Big(f(x)+g(x)\Big) = \frac{d}{dx}\Big(f(x)\Big) + \frac{d}{dx}\Big(g(x)\Big)$.
- The derivative of a sum is the sum of the derivatives.

$\boxed{\textbf{Example 5:}}$ Find the derivative of each of the following functions with respect to x:

(a) $f(x) = x^3 + 2x^2 + \sqrt{x} + 17$

$\sqrt{x} = x^{1/2}$

$$f'(x) = (x^3)' + (2x^2)' + (\sqrt{x})' + (17)'$$

$$= 3x^{3-1} + 2\cdot 2x^{2-1} + \frac{1}{2}\cdot x^{-1/2} + 0$$

$\Rightarrow f'(x) = 3x^2 + 4x + \frac{1}{2}x^{-1/2}$

(b) $y = \dfrac{x^2+x^7}{x^5}$ ← Simplify first!

$$y = \frac{x^2}{x^5} + \frac{x^7}{x^5} = x^{-3} + x^2, \text{ so}$$

$$y' = (x^{-3})' + (x^2)' = -3x^{-3-1} + 2x^{2-1}$$

$$= -3x^{-4} + 2x$$

▶ $\boxed{\textbf{The difference rule:}}$ Let $f(x)$ and $g(x)$ be differentiable functions.

- $(f(x)-g(x))' = f'(x)-g'(x)$.
- $\frac{d}{dx}\Big(f(x)-g(x)\Big) = \frac{d}{dx}\Big(f(x)\Big) - \frac{d}{dx}\Big(g(x)\Big)$.
- The derivative of a difference is the difference of the derivatives.

$\boxed{\textbf{Example 6:}}$ Find an equation for the tangent line to the graph of $k(x) = 4x^3 - 7x^2$ at $x = 1$.

Point: $(1, k(1)) = (1, 4\cdot 1^3 - 7\cdot 1^2) = (1, -3)$

Slope: $k'(1) = 4\cdot 3\cdot 1^{3-1} - 7\cdot 2\cdot 1^{2-1} = -2$

Tangent line (in point slope form): $y - (-3) = -2(x-1)$

$y + 3 = -2(x-1)$

written in slope-intercept form:

$$y = -2x - 1$$

$\boxed{\textbf{Note:}}$ The next two results describe formulas for derivatives of products and quotients. These formulas are <u>more complicated</u> than those for sums and differences.

► **The product rule (aka, Leibniz rule):** Let $f(x)$ and $g(x)$ be differentiable functions.

- $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.
- $\frac{d}{dx}\Big(f(x)g(x)\Big) = \frac{d}{dx}\Big(f(x)\Big)\,g(x) + f(x)\,\frac{d}{dx}\Big(g(x)\Big)$.
- The derivative of a product equals the derivative of the first factor times the second one plus the first factor times the derivative of the second one.

Proof: Set $F(x) = f(x)g(x)$. Then,

$$\begin{aligned}
\underline{F'(x)} &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)\boxed{-f(x+h)g(x)+f(x+h)g(x)}-f(x)g(x)}{h} \\
&= \lim_{h\to 0}\left(f(x+h)\frac{g(x+h)-g(x)}{h}+g(x)\frac{f(x+h)-f(x)}{h}\right) \\
&= \left(\lim_{h\to 0}f(x+h)\right)\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)+g(x)\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right) \\
&\overset{\text{continuity}}{=} f(x)\,g'(x)+g(x)\,f'(x) \\
&= \underline{f'(x)\,g(x)+f(x)\,g'(x)}.
\end{aligned}$$

Note: For more details about the proof, please consult page 91 of our textbook.

Example 7: Differentiate with respect to x the function $y = (2x+1)(x^2+2)$.

$$\begin{aligned}
y' &= (2x+1)'\cdot(x^2+2) + (2x+1)\cdot(x^2+2)' \\
&= (2\cdot 1+0)\cdot(x^2+2) + (2x+1)\cdot(2x^{2-1}+0) \\
&= 2(x^2+2) + (2x+1)(2x) \\
&= 6x^2+2x+4
\end{aligned}$$

Recall: this is just another notation for $F'(3)$

Example 8: Suppose $h(x) = x^2+3x+2$, $g(3) = 8$, $g'(3) = -2$, and $F(x) = g(x)h(x)$. Find $\left.\frac{dF}{dx}\right|_{x=3}$.

$$\frac{dF}{dx} = \left(\frac{dg}{dx}\right)h(x) + g(x)\left(\frac{dh}{dx}\right) = g'(x)\cdot h(x) + g(x)\cdot h'(x)$$

Now $\frac{dh}{dx} = (x^2+3x+2)' = 2x+3$. Evaluating at $x=3$,

$$\left.\frac{dF}{dx}\right|_{x=3} = g'(3)h(3) + g(3)h'(3) = (-2)\cdot(3^2+3\cdot 3+2) + 8\cdot(2\cdot 3+3)$$
$$= 32$$

Note: The statement in words of the following theorem sounds too stiff if we use 'numerator' and 'denominator,' so we replace these terms simply with 'top' and 'bottom.'

► **The quotient rule:** Let $f(x)$ and $g(x)$ be differentiable functions.

- $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$.
- $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\dfrac{d}{dx}\Big(f(x)\Big)g(x) - f(x)\dfrac{d}{dx}\Big(g(x)\Big)}{[g(x)]^2}$.
- The derivative of a quotient equals the derivative of the top times the bottom minus the top times the derivative of the bottom, all over the bottom squared.

Note: The quotient rule can be proved in two steps. First, as a special case, one can get a formula for the derivative of the function $\dfrac{1}{g(x)}$ using the limit definition of the derivative. Secondly, one can use the product rule on $\dfrac{f(x)}{g(x)} = f(x)\dfrac{1}{g(x)}$.

Example 9: Differentiate with respect to s the function $g(s) = \dfrac{2s+1}{5-3s}$.

$$g'(s) = \frac{(5-3s)(2s+1)' - (2s+1)(5-3s)'}{(5-3s)^2} = \frac{(5-3s)\cdot(2) - (2s+1)(-3)}{(5-3s)^2}$$

$$= \frac{10-6s+6s+3}{(5-3s)^2} = \frac{13}{(5-3s)^2}$$

Example 10: Suppose $T(x) = 3x+8$, $B(2) = 3$, $\left.\dfrac{dB}{dx}\right|_{x=2} = -2$, and $Q(x) = \dfrac{T(x)}{B(x)}$. Find $Q'(2)$

$$Q'(2) = \frac{B(2)\cdot T'(2) - T(2)\,B'(2)}{B(2)^2}.$$

Now, $B(2) = 3$, $B'(2) = -2$, $T'(x) = (3x+8)' = 3$, so $T'(2) = 3$.
$T(2) = 3\cdot 2 + 8 = 14$, and

$$Q'(2) = \frac{3\cdot 3 - 14(-2)}{3^2} = \frac{37}{9}.$$

Example 11: Find an equation of the tangent line to the graph of $y = \dfrac{4x}{x^2+1}$ at the point $x = 2$.

Point: $y(2) = \dfrac{4 \cdot 2}{2^2+1} = \dfrac{8}{5}$

Slope: y' at $x = 2$. But

$$y' = \frac{(x^2+1)(4x)' - (4x)(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{(x^2+1)\cdot 4 - (4x)(2x)}{(x^2+1)^2}$$

$$= \frac{4 - 4x^2}{(x^2+1)^2}$$

$$y'(2) = \frac{4 - 4\cdot 2^2}{(2^2+1)^2} = -\frac{12}{25}$$

Tangent Line:

$$\boxed{y - \frac{8}{5} = \frac{-12}{25}(x-2)}$$

Example 12: Suppose that the equation of the tangent line to the graph of $g(x)$ at $x = 9$ is given by the equation

$$y = 21 + 2(x - 9)$$

Find $g(9)$ and $g'(9)$.

$\begin{pmatrix}\text{Tangent line} \\ \text{at } x = 9\end{pmatrix}$: $y - g(9) = g'(9)(x-9)$

$\Rightarrow y = \underset{21}{g(9)} + \underset{2}{g'(9)}(x-9)$ so $g(9) = 21$, $g'(9) = 2$.

Point on $g(x)$: $(2, g(2)) = (2, 5\cdot 2 + f(2)) = (2, 10+3) = (2, 13)$

Slope of $g(x)$: $g'(x) = (5x)' + f'(x) = 5 + f'(x)$

$g'(2) = 5 + f'(2) = 5 + (-2) = 3$

Example 13: So $y - 13 = 3(x-2) \Rightarrow y = 13 + 3x - 6$
$= 3x + 7$

A segment of the tangent line to the graph of $f(x)$ at x is shown in the picture. Using information from the graph we can estimate that

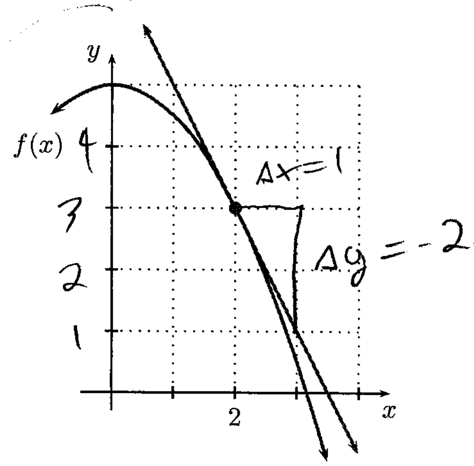
$f(2) =$ 3 $\quad f'(2) = \dfrac{\Delta y}{\Delta x} = -2$

hence the equation to the tangent line to the graph of

$$g(x) = 5x + f(x)$$

at $x = 2$ can be written in the form $y = mx + b$ where

$m =$ 3 $\quad b =$ 7



▶ **Composite functions:** A function $h(x)$ is said to be a composite function of $f(x)$ followed by $g(x)$ if $h(x) = (g \circ f)(x) = g(f(x))$. We may write: $h : x \overset{f}{\longmapsto}$ ________ $\overset{g}{\longmapsto}$ ________.

Example 14: Find functions $f(x)$ and $g(x)$, not equal x, such that $h(x) = g(f(x))$:

(a) $h(x) = (x^4 + 2x^2 + 7)^{21}$ $\qquad h : x \overset{f}{\longmapsto} \underline{x^4+2x^2+7} \overset{g}{\longmapsto} \underline{(x^4+2x^2+7)^{21}}$

Ans: $f(x) \overset{?}{=} \underline{x^4+2x^2+7}$ and $g(x) \overset{?}{=} \underline{x^{21}}$

(b) $h(x) = \sqrt{x^3 - 3x + 1}$ $\qquad h : x \overset{f}{\longmapsto} \underline{x^3-3x+1} \overset{g}{\longmapsto} \underline{(x^3-3x+1)^{1/2}}$

Ans: $f(x) \overset{?}{=} \underline{x^3-3x+1}$ and $g(x) \overset{?}{=} \underline{x^{1/2}}$

Note: In a SMS (short message service) competition for the title of "Fastest SMS Thumbs," it is observed that Competitor A inputs text three times faster than B, and Competitor B inputs text two times faster than C. How much faster is Competitor A than Competitor C? Why?

A —— rate of A relative to B (3) —— B —— rate of B relative to C (2) —— C

¿ rate of A relative to C ?

Suppose $y = g(f(x))$. To find a formula for $\dfrac{dy}{dx} = \dfrac{d}{dx}[g(f(x))]$, we set $u = f(x)$ so that $y = g(u)$.

y —— rate of y relative to u $\left(\dfrac{dy}{du}\right)$ —— u —— rate of u relative to x $\left(\dfrac{du}{dx}\right)$ —— x

¿ rate of y relative to x ?

We expect:

$$\boxed{\frac{dy}{dx} = \qquad}$$

Our guess is in fact correct, and the formula for $\dfrac{dy}{dx}$ is called the **chain rule** (in Leibniz notation).

▶ **The chain rule:**

Let $f(x)$ and $g(x)$ be functions, with f differentiable at x and g differentiable at the point $f(x)$. We have:

- $(g(f(x)))' = g'(f(x))f'(x)$.
- Let $y = g(u)$ and $u = f(x)$. Then $y = g(u) = g(f(x))$ and
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$
- The derivative of a composite function equals the derivative of the outside function, evaluated at the inside part, times the derivative of the inside part.

Note: A special case of the chain rule is called the **power chain rule**.

If $\quad y = [f(x)]^n \quad$ then $\quad \dfrac{dy}{dx} = n\,[f(x)]^{n-1} \cdot f'(x)$.

Example 15:

(a) Suppose $k(x) = (1+3x^2)^3$. Find $k'(x)$.

$k(x) = (\text{Stuff})^3$, so Chain Rule $\Rightarrow k'(x) = 3\cdot(\text{Stuff})^{3-1}\cdot \text{Stuff}'$

Now fill in "stuff" $\Longrightarrow k'(x) = 3\cdot(1+3x^2)^2\cdot(1+3x^2)'$

$= 3(1+3x^2)^2\cdot 6x = \boxed{18x\,(1+3x^2)^2}$

(b) Suppose $g(s) = (s^3 - 4s^2 + 12)^5$. Find $\dfrac{dg}{ds}$.

$g(s) = (\text{Stuff})^5$, so $\dfrac{dg}{ds} = 5(\text{Stuff})^{5-1}\cdot \text{Stuff}'$

Fill in "stuff" $\boxed{\dfrac{dg}{ds} = 5\cdot(s^3-4s^2+12)^4\cdot(3s^2-8s)}$

Example 16: Differentiate the following functions with respect to the appropriate variable:

Rewrite these 4 simplify first.

(a) $f(s) = \dfrac{1}{\sqrt[4]{5s-3}} = (5s-3)^{-1/4}$

So $f'(s) = \left(-\frac{1}{4}\right)(5s-3)^{-\frac{1}{4}-1}\cdot(5s-3)'$

$= \left(-\frac{1}{4}\right)\cdot(5s-3)^{-5/4}\cdot 5 = \boxed{\left(-\frac{5}{4}\right)(5s-3)^{-5/4}}$

(b) $g(t) = \sqrt{t^2+7} = (t^2+7)^{1/2}$

$g'(t) = \frac{1}{2}(t^2+7)^{\frac{1}{2}-1}\cdot(t^2+7)' = \frac{1}{2}\cdot(t^2+7)^{-1/2}\cdot 2t$

$= \boxed{t\,(t^2+7)^{-1/2}}$

Provided $x > 4$.

(c) $h(x) = \dfrac{\sqrt{x^2-16}}{\sqrt{x-4}} = \dfrac{\sqrt{(x-4)(x+4)}}{\sqrt{x-4}} \overset{}{=} \dfrac{\cancel{\sqrt{x-4}}\sqrt{x+4}}{\cancel{\sqrt{x-4}}} = \sqrt{x+4}$

So $h'(x) = \left(\sqrt{x+4}\right)' = \left((x+4)^{1/2}\right)' = \frac{1}{2}(x+4)^{-1/2}\cdot(x+4)'$

$\boxed{= \frac{1}{2}(x+4)^{-1/2}}$

(d) $k(x) = (x^2-3)\sqrt{x-9}$.

Product! $k'(x) = (x^2-3)'\cdot(x-9)^{1/2} + (x^2-3)\left((x-9)^{1/2}\right)'$

$= (2x)(x-9)^{1/2} + (x^2-3)\cdot\left(\frac{1}{2}\right)(x-9)^{-1/2}\cdot(x-9)'$

$= \boxed{2x\,(x-9)^{1/2} + \frac{1}{2}(x^2-3)(x-9)^{-1/2}}$

Chain Rule!

Example 17: Suppose $F(x) = g(h(x))$.
If $h(2) = 7$, $h'(2) = 3$, $g(2) = 9$, $g'(2) = 4$, $g(7) = 5$ and $g'(7) = 11$, find $F'(2)$.

$F'(x) = g'(h(x)) \cdot h'(x)$, so $F'(2) = g'(h(2)) \cdot h'(2)$ Plug in $h(2)=7$, $h'(2)=3$

$= g'(7) \cdot 3$

$= 11 \cdot 3$ Now use $g'(7)=11$

$= 33.$

Example 18: Suppose $g(x) = f(x^2 + 3(x-1) + 5)$ and $f'(6) = 21$. Find $g'(1)$.
(Note: $f(x^2 + 3(x-1) + 5)$ means "the function f, applied to $x^2 + 3(x-1) + 5$," not "a number f multiplied with $x^2 + 3(x-1) + 5$.")

$g(x) = f(\text{stuff})$, so $g'(x) = f'(\text{stuff}) \cdot \text{stuff}'$

$= f'(x^2 + 3(x-1) + 5) \cdot (x^2 + 3(x-1)+5)'$

$= f'(x^2 + 3(x-1) + 5) \cdot (2x + 3)$

so $g'(1) = f'(1^2 + 3(1-1) + 5) \cdot (2 \cdot 1 + 3)$

$= f'(6) \cdot 5 = 21 \cdot 5 = 105.$

Example 19: Suppose $h(x) = \sqrt{f(x)}$ and the equation of the tangent line to $f(x)$ at $x = -1$ is $y = 9 + 3(x+1)$. Find $h'(-1)$.

$h(x) = f(x)^{1/2}$, so $h'(x) = \frac{1}{2} \cdot f(x)^{1-\frac{1}{2}} f'(x) = \frac{1}{2} f(x)^{-1/2} f'(x)$

$= \dfrac{f'(x)}{2\sqrt{f(x)}}$

so $h'(-1) = \dfrac{f'(-1)}{2\sqrt{f(-1)}}$. Now $y = 9 + 3(x+1)$ tangent to $y = f(x)$
so $9 = f(-1)$, $3 = f'(-1)$.

so $h'(-1) = \dfrac{3}{2\sqrt{9}} = \dfrac{1}{2}$

Example 20: Suppose $\boxed{F(G(x)) = x^2}$ and $G'(1) = 4$. Find $F'(G(1))$.

Compute derivative on both sides

$F(G(x))' = (x^2)'$

$F'(G(x)) \cdot G'(x) = 2x$ Plug in $x=1$ $\Longrightarrow$ $F'(G(1)) \cdot \overbrace{G'(1)}^{4} = 2 \cdot 1$

(Chain Rule) (Power Rule)

so $4 \cdot F'(G(1)) = 2$

so $\boxed{F'(G(1)) = \frac{2}{4} = \frac{1}{2}.}$

► $\boxed{\textbf{Higher Derivatives}}$

Let $y = f(x)$ be a differentiable function and $f'(x)$ its derivative. If $f'(x)$ is again differentiable, we write

$$y'' = f''(x) = (f'(x))'$$

and call it the *second derivative* of $f(x)$.

In Leibniz notation:

$$\frac{d^2}{dx^2}\Big(f(x)\Big) \qquad \text{or} \qquad \frac{d^2y}{dx^2}$$

Similarly, we can define higher derivatives of $f(x)$ if they exist. For example, the third derivative $f'''(x)$ of $f(x)$ is the derivative of $f''(x)$, etc. The higher derivatives are denoted $f^{(4)}(x)$, $f^{(5)}(x)$, and so on.

Example 21: Let $H(s) = s^5 - 2s^3 + 5s + 3$. Find $H''(s)$.

$$H'(s) = 5s^4 - 6s^2 + 5$$

$$\boxed{H''(s) = 20s^3 - 12s}$$

Example 22: Let $f(x) = \dfrac{2x+1}{x+1}$. Find $\dfrac{d^2f}{dx^2}$.

$$\frac{df}{dx} = \frac{(x+1)(2x+1)' - (2x+1)(x+1)'}{(x+1)^2} = \frac{(x+1)\cdot 2 - 2x - 1}{(x+1)^2} = \frac{1}{(x+1)^2}$$

$$= (x+1)^{-2}$$

$$\text{so } \frac{d^2f}{dx^2} = \frac{d}{dx}\left((x+1)^{-2}\right) = -2(x+1)^{-2-1}(x+1)'$$

$$= -2(x+1)^{-3}$$

Example 23: Let $f(x) = \sqrt{x}$. Find the third derivative, $f^{(3)}(x)$.

$$f(x) = x^{1/2}$$

$$f^{(1)}(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2}$$

$$f^{(2)}(x) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\cdot x^{-\frac{1}{2}-1} = \left(-\frac{1}{4}\right)x^{-3/2}$$

$$f^{(3)}(x) = \left(-\frac{1}{4}\right)\left(-\frac{3}{2}\right)x^{-\frac{3}{2}-1} = \boxed{\left(\frac{3}{8}\right)x^{-5/2}}$$

$\boxed{\textbf{Example 24:}}$ Let $f(x) = \sqrt{x^3} + \sqrt{x}$. Find $\dfrac{df}{dx}$. $f(x) = x^{\frac{3}{2}} + x^{\frac{1}{2}}$

So $\dfrac{df}{dx} = \dfrac{3}{2}x^{\frac{3}{2}-1} + \dfrac{1}{2}x^{\frac{1}{2}-1}$

$= \dfrac{3}{2}x^{1/2} + \dfrac{1}{2}x^{-1/2}$

$\boxed{\textbf{Example 25:}}$ If $f(x) = (7x-13)^3$, find $f''(x)$.

$f(x) = (\text{Stuff})^3$ so $f'(x) = 3(7x-13)^2 \cdot 7 = 21(7x-13)^2$

Notice $f''(x)$ will require chain rule <u>again</u>

$f''(x) = 21 \cdot ((7x-13)^2)' = 21 \cdot 2 \cdot (7x-13)^1 \cdot 7 = 294(7x-13)$

$\boxed{\textbf{Example 26:}}$ If $f(x) = x^4$, find $f^{(5)}(x)$, the 5th derivative of $f(x)$. Can you make a guess about the $(n+1)$st derivative of $f(x) = x^n$.

$\dfrac{d}{dx}(x^4) = 4x^3$, $\dfrac{d^2}{dx^2}(x^4) = \dfrac{d}{dx}(4x^3) = 4 \cdot 3x^2$, $\dfrac{d^3}{dx^3}(x^4) = \cdots = 4 \cdot 3 \cdot 2 \cdot x$

$\dfrac{d^4}{dx^4}(x^4) = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, so $\dfrac{d^5}{dx^5}(x^4) = (24)' = 0$

In general, if $f(x) = x^n$, and $m > n$, then $f^{(m)}(x) = 0$

▶ $\boxed{\textbf{Acceleration:}}$ Suppose $s(t)$ measures the position (or height) of an object from a given point at time t. We recall that the derivative of $s(t)$ is the velocity $v(t) = s'(t)$ of the object. Now, the *acceleration* of an object measures the rate of change of the velocity of the object with repect to time:

$$\text{Acceleration} = v'(t) = \frac{dv}{dt} \qquad \text{but} \qquad \text{Velocity} = s'(t) = \frac{ds}{dt} \qquad \text{so} \qquad \text{Acceleration} = s''(t) = \frac{d^2s}{dt^2}$$

The acceleration is usually denoted by $a(t)$. Thus, $a(t) = s''(t)$.

$\boxed{\textbf{Example 27:}}$ Suppose the height in feet of an object above ground at time t (in seconds) is given by

$$h(t) = -16t^2 + 12t + 200$$

Find the acceleration of the object after 3 seconds.

$a(t) = v'(t)$, But $v(t) = h'(t) = -32t + 12$

so $a(t) = (-32t+12)' = -32$.