## MA123, Chapter 6: Extreme values, Mean Value Theorem, Curve sketching, and Concavity

| Chapter Goals: | • Apply the Extreme Value Theorem to find the global extrema for continuous function on closed and bounded interval. |
|---|---|
| | • Understand the connection between critical points and local extreme values. |
| | • Understand the relationship between the sign of the derivative and the intervals on which a function is increasing and on which it is decreasing. |
| | • Understand the statement and consequences of the Mean Value Theorem. |
| | • Understand how the derivative can help you sketch the graph of a function. |
| | • Understand how to use the derivative to find the global extreme values (if any) of a continuous function over an unbounded interval. |
| | • Understand the connection between the sign of the second derivative of a function and the concavities of the graph of the function. |
| | • Understand the meaning of inflection points and how to locate them. |

| Assignments: | Assignment 12 | Assignment 13 |
|---|---|---|
| | Assignment 14 | Assignment 15 |

Finding the largest profit, or the smallest possible cost, or the shortest possible time for performing a given procedure or task, or figuring out how to perform a task most productively under a given budget and time schedule are some examples of practical real-world applications of Calculus. The basic mathematical question underlying such applied problems is how to find (if they exist) the largest or smallest values of a given function on a given interval. This procedure depends on the nature of the interval.

▶ **Global (or absolute) extreme values:**   The largest value a function (possibly) attains on an interval is called its **global (or absolute) maximum value**. The smallest value a function (possibly) attains on an interval is called its **global (or absolute) minimum value**. Both maximum and minimum values (if they exist) are called **global (or absolute) extreme values**.

**Example 1(a):**
Find the maximum and minimum values for the function
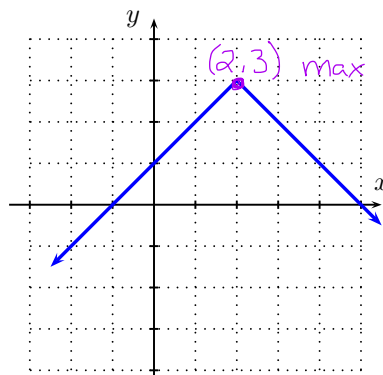
$$f(x) = (x-1)^2 - 3,$$

if they exist.

**Example 1(b):**
Find the maximum and minimum values for the function

$$f(x) = -|x-2| + 3,$$

if they exist.

**Example 1(c):**
Find the maximum and minimum values for the function

$$f(x) = x^2 + 1, \qquad x \in [-1, 2]$$
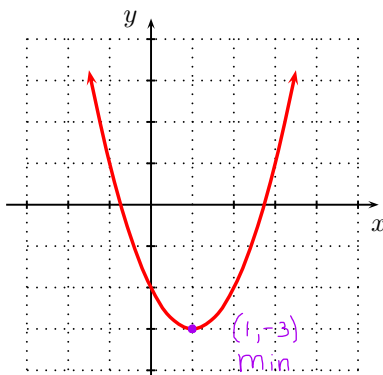
if they exist.



−3 is the absolute minimum value and this occurs at x=1. No absolute maximum value

3 is the absolute maximum value and this occurs at x=2. No absolute minimum value
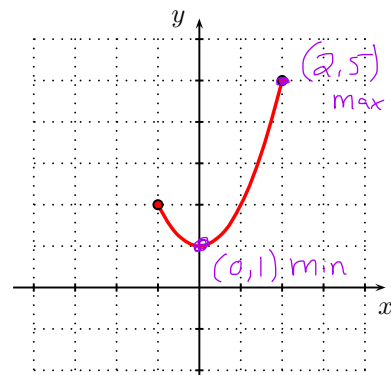
5 is the absolute maximum value and this occurs at x=2. 1 is the absolute minimum value and this occurs at x=0

61

We first focus on continuous functions on a closed and bounded interval. The question of largest and smallest values of a continuous function $f$ on an interval that is not closed and bounded requires us to pay more attention to the behavior of the graph of $f$, and specifically to where the graph is rising and where it is falling.

### Closed and bounded intervals:

An interval is **closed and bounded** if it has finite length and contains its endpoints.

For example, the interval $[-2, 5]$ is closed and bounded.
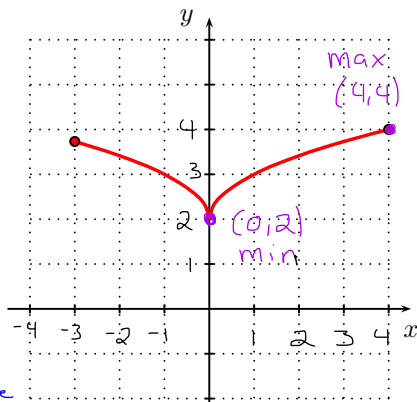
▶ ### The Extreme Value Theorem (EVT):

If a function $f$ is continuous on a closed, bounded interval $[a, b]$, then the function $f$ attains a maximum and a minimum value on $[a, b]$.

### Example 2(a): Let $f(x) = \begin{cases} 2 + \sqrt{x} & \text{if } x > 0 \\ 2 + \sqrt{-x} & \text{if } x \le 0. \end{cases}$

Does $f(x)$ have a maximum and a minimum value on $[-3, 4]$? How does this example illustrate the Extreme Value Theorem?

On $[-3, 4]$ $f(x)$ has an absolute maximum of 4 at $x = 4$ and an absolute minimum of 2 at $x = 0$.
Note: $[-3, 4]$ is a closed and bounded interval and from the graph one sees that $f$ is continuous on this interval; therefore, by the EVT $f$ must have an absolute maximum and minimum on $[-3, 4]$.

### Example 2(b): Let $g(x) = \dfrac{1}{x}$. Does $g(x)$ have a maximum value

and a minimum value on $[-2, 3]$? Does this example contradict the Extreme Value Theorem? Why or why not?

On $[-2, 3]$ $g(x)$ does not have an absolute maximum or minimum value. However, this does not contradict the EVT because $g(x)$ is not continuous on $[-2, 3]$ since $g(x)$ is discontinuous (undefined) at $x = 0 \in [-2, 3]$.
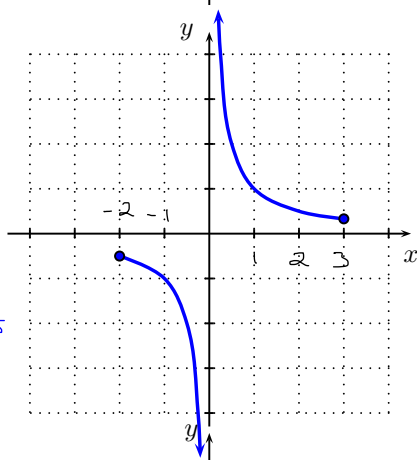
### Example 2(c): Let $h(x) = x^4 - 2x^2 + 1$. Does $h(x)$ have a

maximum value and a minimum value on $(-1.25, 1.5)$? Does this example contradict the Extreme Value Theorem? Why or why not?

On $(-1.25, 1.5)$ $h(x)$ has an absolute minimum of 0 at $x = -1$ and $x = 1$; however, $h(x)$ has no absolute maximum on this interval. This does not contradict the EVT because $(-1.25, 1.5)$ is not a closed interval.

62
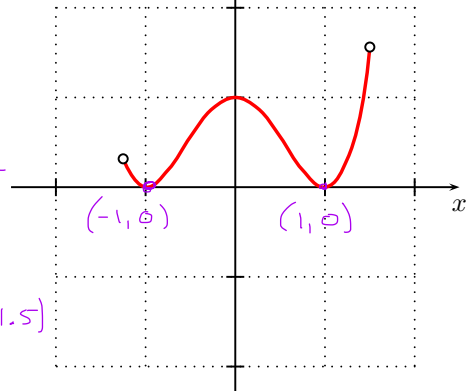
The EVT is an existence statement; it doesn't tell you how to locate the maximum and minimum values of $f$.

The following results tell you how to narrow down the list of possible points on the given interval where the function $f$ *might* have an extreme value to (usually) just a few possibilities. You can then evaluate $f$ at these few possibilities, and pick out the smallest and largest value.

▶ **Fermat's Theorem:** Let $f(x)$ be a continuous function on the interval $[a,b]$. If $f$ has an extreme value at a point $c$ strictly between $a$ and $b$, and if $f$ is differentiable at $x=c$, then $f'(c) = 0$.

▶ **Corollary:** Let $f(x)$ be a continuous function on the closed, bounded interval $[a,b]$. If $f$ has an extreme value at $x=c$ in the interval, then either
- $c=a$ or $c=b$;
- $a < c < b$ and $f'(c)=0$;
- $a < c < b$ and $f$ is not differentiable at $x=c$, so that $f'$ is not defined at $x=c$.

**Example 3:** Find the maximum and minimum values of $f(x) = x^3 - 3x^2 - 9x + 5$ on the interval $[0,4]$. $\quad$ continuous and differentiable everywhere

For which values $x$ are the maximum and minimum values attained? min @ $x=3$ & max @ $x=0$

$f'(x) = 3x^2 - 6x - 9$ ← defined everywhere

Need to solve $\quad 0 = 3x^2 - 6x - 9$

$0 = x^2 - 2x - 3$

$0 = (x-3)(x+1)$

$x - 3 = 0 \quad$ or $\quad x+1=0$
$\quad x = 3 \qquad\qquad x = -1$

To find max/min values on $[0,4]$ must check $\quad$ Notation $-1 \notin (0,4)$
- Endpoints, $x=0$ & $x=4$
- $x$ in $(0,4)$ where $f'(x)=0$, $x=3$ (note $-1$ is **not** in $(0,4)$)
- $x$ in $(0,4)$ where $f'(x)$ DNE, none

$f(0) = 0^3 - 3(0)^2 - 9(0) + 5 = \boxed{5}$ ← absolute maximum Value
$f(3) = 3^3 - 3(3)^2 - 9(3) + 5 = 27 - 27 - 27 + 5 = -27+5 = \boxed{-22}$ ← absolute minimum value
$f(4) = 4^3 - 3(4)^2 - 9(4) + 5 = 64 - 48 - 36 + 5 = 16 - 36 + 5 = -20 + 5 = -15$

**Example 4:** Find the maximum and minimum values of $F(s) = \dfrac{2s+1}{s-6}$ on the interval $[-1,5]$. For which values $s$ are the maximum and minimum values attained?

$F'(s) = \dfrac{(s-6)(2) - (2s+1)(1)}{(s-6)^2}$

$= \dfrac{2s - 12 - 2s - 1}{(s-6)^2}$

$= \dfrac{-13}{(s-6)^2}$ ← Never zero. Undefined at $x=6$

Never zero
$\dfrac{-13}{(s-6)^2} = 0$

$-13 = 0 (s-6)^2$
$-13 = 0 \therefore$ so $F'(s) \neq 0$

$\boxed{\begin{array}{l} \text{min @ } s=5 \\ \text{max @ } s = -1 \end{array}}$

To find max/min values on $[-1,5]$ must check
- Endpoints, $s=-1$ & $s=5$
- $x$ in $(-1,5)$ where $F'(s)=0$, nowhere
- $x$ in $(-1,5)$ where $F'(x)$ DNE, nowhere (note $6 \notin (-1,5)$)

$F(-1) = \dfrac{2(-1)+1}{-1-6} = \dfrac{-2+1}{-1-6} = \dfrac{-1}{-7} = \boxed{\dfrac{1}{7}}$ ← absolute maximum Value

$F(5) = \dfrac{2(5)+1}{5-6} = \dfrac{10+1}{5-6} = \dfrac{11}{-1} = \boxed{-11}$ ← absolute minimum Value

**Example 5:** Find the maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-1,8]$. For which values $s$ are the maximum and minimum values attained?

$f'(x) = \dfrac{2}{3} x^{-1/3} = \dfrac{2}{3\sqrt[3]{x}}$ ← Never zero. Undefined at $x=0$

Never zero
$\dfrac{2}{3\sqrt[3]{x}} = 0 \quad \cancel{2=0}$ so $f'(x) \neq 0$
$2 = 0 \cdot (3\sqrt[3]{x})$

To find max/min values on $[-1,8]$ must check
- Endpoints, $x=-1$ and $x=8$
- $x$ in $(-1,8)$ where $f'(x)=0$, no where
- $x$ in $(-1,8)$ where $f'(x)$ DNE, $x=0$

$\boxed{\begin{array}{l} \text{min @ } x=0 \\ \text{max @ } x=8 \end{array}}$

$f(-1) = (-1)^{2/3} = \sqrt[3]{(-1)^2} = \sqrt[3]{1} = 1$

$f(0) = 0^{2/3} = \sqrt[3]{0^2} = \sqrt[3]{0} = \boxed{0}$ ← Absolute minimum

and $f(8) = 8^{2/3} = (2^3)^{2/3} = 2^2 = \boxed{4}$ ← absolute maximum

**Example 6:** Find the $t$ values on the interval $[-10, 10]$ where $g(t) = |t - 4| + 7$ takes its maximum and minimum values. What are the maximum and minimum values?


$y = |t|$
$y = |t - 4|$ (4,0)
$g(t) = |t - 4| + 7$ (4,7)
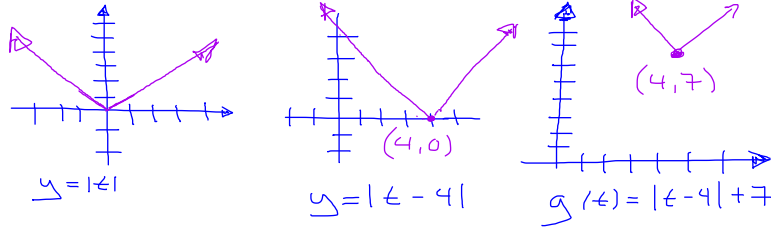
Geometrically one sees that $g(t)$ has an absolute minimum value of 7 and this occurs at $x = 4$. Moreover, the graph of $g(t)$ show the absolute maximum value on $[-10, 10]$ must occur at one of the endpoints $x = -10$ and $x = 10$.

$g(-10) = |-10-4| + 7 = |-14| + 7$
$= 14 + 7 = \boxed{21} \leftarrow$ absolute maximum

$g(10) = |10-4| + 7 = |6| + 7$
$= 6 + 7 = 13$

**Example 7:** Find the maximum and minimum values of $k(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \leq 1 \\ -3x + 7 & \text{if } x > 1 \end{cases}$ on the interval $[-2, 3]$.

Note $k'(x) = \begin{cases} 2x + 2 & \text{if } x < 1 \\ -3 & \text{if } x > 1 \end{cases}$

at $x = 1$ $k'(x)$ is undefined because

$k'(1) = \lim_{h \to 0} \frac{k(1+h) - k(1)}{h}$ ← $k(1) = 1^2 + 2(1) + 1$
$= 1 + 2 + 1$
$= 4$

$\lim_{h \to 0^-} \frac{k(1+h) - k(1)}{h} = \lim_{h \to 0^-} \frac{(1+h)^2 + 2(1+h) + 1 - 4}{h} = 4 \neq \lim_{h \to 0^+} \frac{k(1+h) - k(1)}{h} = \lim_{h \to 0^+} \frac{-3(1+h) + 7 - 4}{h} = -3$

To find max/min on $[-2, 3]$ must check
• Endpoints, $x = -2$ and $x = 3$
• $x$ in $(-2, 3)$ where $k'(x) = 0$, $x = -1$
• $x$ in $(-2, 3)$ where $k'(x)$ DNE, $x = 1$

Also, Note $k'(x) = 0$ when
$2x + 2 = 0$ so $x = -1$
$2x = -2$

Need $k(x)$ continuous on $[-2, 3]$ only questionable at $x = 1$; however,
$\lim_{x \to 1} k(x) = 4 = k(1)$
$\lim_{x \to 1^+} x^2 + 2x + 1 = 4 = \lim_{x \to 1^-} -3x + 7$

**Example 8:** Find the maximum and minimum values of $g(x) = 1 + x + x^2 + x^3$ on the interval $[0, 2]$. For which values $x$ are the maximum and minimum values attained? ← continuous everywhere

$g'(x) = 1 + 2x + 3x^2$
↑ Defined everywhere

Need to solve $0 = 1 + 2x + 3x^2$

Discriminant $= 2^2 - 4(3)(1)$
$= 4 - 12 = -8 < 0$

So no real solution
So $g'(x)$ is never zero

To find max/min on $[0, 2]$ must check
• Endpoints, $x = 0$ and $x = 2$
• $x$ in $(0, 2)$ where $g'(x) = 0$, nowhere
• $x$ in $(0, 2)$ where $g'(x)$ DNE, nowhere

$g(0) = 1 + 0 + 0^2 + 0^3 = \boxed{1} \leftarrow$ absolute minimum value
$g(2) = 1 + 2 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = \boxed{15} \leftarrow$ absolute maximum value

min @ $x = 0$
max @ $x = 2$

▶ **Local (or relative) extreme points:** In addition to the points where a function might have a maximum or minimum value, there are other points that are important for the behavior of the function and the shape of its graph.



global max
local max
local max
local min
local max
local min
global min
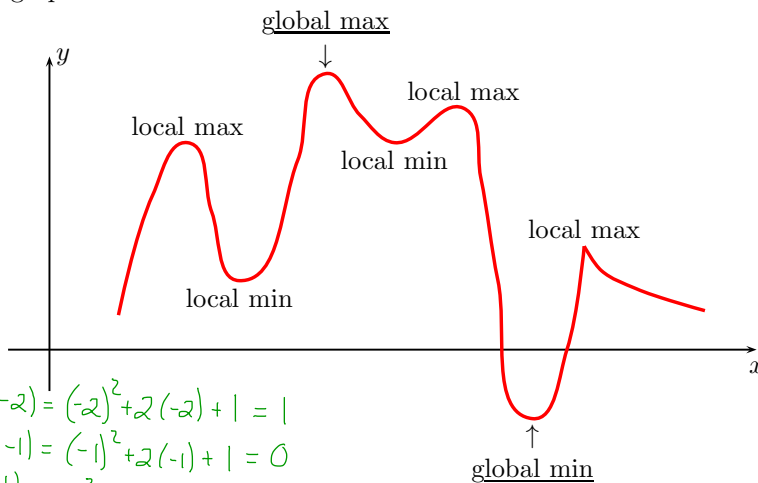
If you think of the graph of the function as the profile of a landscape, the global maximum could represent the highest hill in the landscape, while the minimum could represent the deepest valley.

The other points indicated in the graph, which look like tops of hills (although not the highest hills) and bottom of valleys (although not the deepest valleys), are called **local** (or **relative**) **extreme values**.

$k(-2) = (-2)^2 + 2(-2) + 1 = 1$
$k(-1) = (-1)^2 + 2(-1) + 1 = 0$
$k(1) = 1^2 + 2(1) + 1 = \boxed{4} \leftarrow$ absolute maximum
$k(3) = -3(3) + 7 = \boxed{-2} \leftarrow$ absolute minimum

64

**Definition:** A function $f$ has a **local** (or **relative**) **maximum** at a point $(c, f(c))$ if there is some interval about $c$ such that $f(c) \geq f(x)$ for all $x$ in that interval. A function $f$ has a **local** (or **relative**) **minimum** at a point $(c, f(c))$ if there is some interval about $c$ such that $f(c) \leq f(x)$ for all $x$ in that interval.

**Theorem:** If $f$ has a local extreme value at $(c, f(c))$ and is differentiable at that point $c$, then $f'(c) = 0$

**Critical points:** Let $f$ be a function. If $f$ is defined at the point $x = c$ and either $f'(c) = 0$ or $f'(c)$ is undefined then the point $c$ is called a **critical point** of $f$.

▶ **Increasing and decreasing functions:** A function $f$ is said to be increasing when its graph rises and decreasing when its graph falls. More precisely, we say that:

$f$ is **increasing** on an interval $I$ if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in $I$.

$f$ is **decreasing** on an interval $I$ if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in $I$.



$f$ is increasing
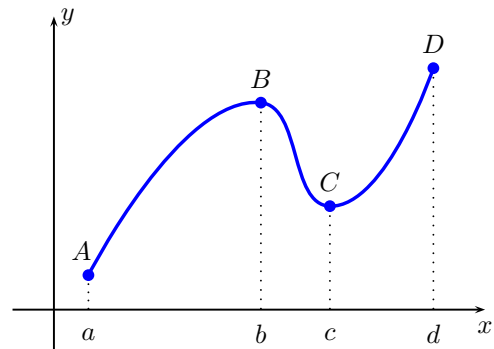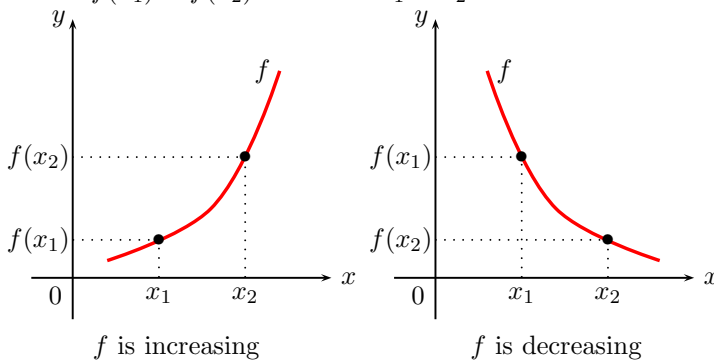
$f$ is decreasing

$f$ is increasing on $[a, b]$ and $[c, d]$
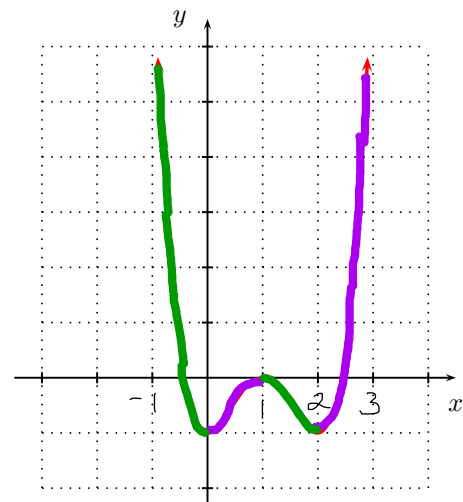$f$ is decreasing on $[b, c]$.

**Example 9:** The picture shows the graph of $y = f(x)$.

(a) Find the intervals on which $f(x)$ is increasing and decreasing.

$f$ is increasing on: $(0, 1) \cup (2, \infty)$
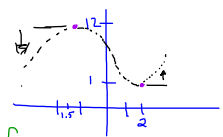$f$ is decreasing on $(-\infty, 0) \cup (1, 2)$

(b) Find the intervals on which the tangent line to $f(x)$ has positive slope, and the intervals on which the tangent line has negative slope.

Positive slope on $(0, 1) \cup (2, \infty)$
Negative slope on $(-\infty, 0) \cup (1, 2)$



65

**Example 10:** Suppose that $f(-1.5) = 12$ and $f(2) = 1$. In addition, you are given that $f(x)$ continuous everywhere, is increasing on the intervals $(-\infty, -1.5] \cup [2, \infty)$ and $f(x)$ is decreasing on the interval $[-1.5, 2]$. Which of the following are **not** possible?

*(handwritten annotations)*

OK  (a) $f(-2) = 7$   Note $-2 \in (-\infty, -1.5]$  False   (e) $f(3) = -1$   $3 \in [2, \infty)$ and $2 < 3$ so $f(2) < f(3)$  False

OK  (b) $f(-2) = 11$   and $-2 < -1.5$ so $f(-2) < f(-1.5) = 12$  OK   (f) $f(4) = 6$   $4 \in [2, \infty)$ and $2 < 4$ so $f(2) < f(4)$ ☺

False  (c) $f(-1) = 13$   7 or 11   $-1 \in [-1.5, 2]$ and   False   (g) $f(3) = 5$ and $f(4) = 4$   $3$ and $4 \in [2, \infty)$ and $3 < 4$ so $f(3) < f(4)$  False

False  (d) $f(0) = 0$   $-1.5 < -1$ so $f(-1.5) > f(-1)$   OK   (h) $f(3) = 5$ and $f(4) = 6$   $3$ and $4 \in [2, \infty)$ and $2 < 3 < 4$ so

$0 \in [-1.5, 2]$ and $0 < 2$ so $0 = f(0) > f(2) = 1$  False     $12 \quad 13$   False     $1 = f(2) < f(3) < f(4) = 6$ ☺

In general, using the algebraic definition of increasing/decreasing to verify that a function is increasing and decreasing can be difficult. However, example 9 suggests an easier method for checking where a function is increasing or decreasing:

- If $f(x)$ is increasing then $f'(x) > 0$;     - If $f(x)$ is decreasing then $f'(x) < 0$.

We would like to be able to reverse these implications, but doing so will require some additional theory.

▶ **The Mean Value Theorem (MVT):**   If $f$ is continuous on $[a, b]$ and differentiable at every point strictly between $a$ and $b$, then there exists some point $x = c$ (and maybe more than one) strictly between $a$ and $b$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

**Geometric interpretation of the MVT:**   For some (non-necessarily unique) point $c$ between $a$ and $b$ the tangent line to the graph of $f$ at $P(c, f(c))$ has the same slope as the secant line connecting the points $A(a, f(a))$ and $B(b, f(b))$ on the graph of $f$.
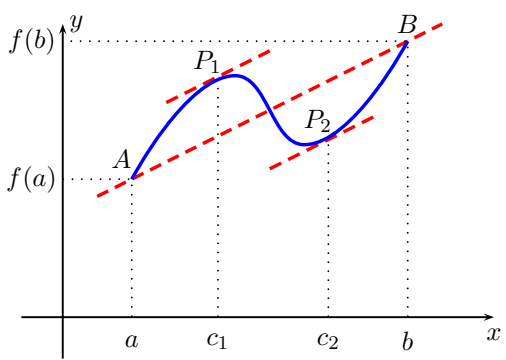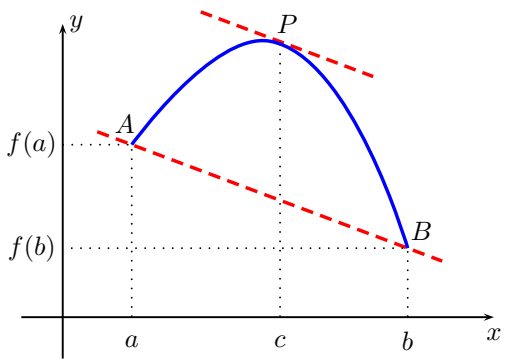


**A rewording of the MVT:**

If $f$ is continuous on $[a, b]$ and differentiable at every point strictly between $a$ and $b$, then there exists some point $x = c$ (and maybe more than one) strictly between $a$ and $b$ such that

| the average rate of change of $f$ on $[a, b]$ | = | the instantaneous rate of change of $f$ at $x = c$. |
|---|---|---|

**Example 11:** Let $Q(t) = t^2$.   Find a value $A \neq 1$ such that the average rate of change of $Q(t)$ from 1 to $A$ equals the instantaneous rate of change of $Q(t)$ at $t = 2$.

*(handwritten work)*

$Q(t) = t^2$

$Q'(t) = 2t$

$Q'(2) = 2 \cdot 2 = 4$

The instantaneous rate of change of $Q$ at $x = 2$

Average Rate of Change of $Q$ on $[1, A]$ $= \dfrac{Q(A) - Q(1)}{A - 1} = \dfrac{A^2 - 1^2}{A - 1} = \dfrac{(A-1)(A+1)}{A-1} = A + 1$

66   So one wants   $A + 1 = 4$   $\boxed{A = 3}$
          $\quad\ -1 \ -1$

**Example 12:** Let $f(x) = x - x^3$. Verify that the function satisfies the hypotheses of the Mean Value Theorem on the interval $[-2, 0]$. Then find all numbers $c$ that satisfy the conclusion of the Mean value Theorem.

*continuous and differentiable on $\mathbb{R}$ so $f$ is continuous on $[-2,0]$ and differentiable on $(-2,0)$*

$f'(x) = 1 - 3x^2$   Now one needs to solve the equation: $1 - 3x^2 = -3$

$$\text{AROC on} \atop [-2,0] = \frac{f(0) - f(-2)}{0 - (-2)} = \frac{[0 - 0^3] - [-2 - (-2)^3]}{0 + 2}$$

$$= \frac{0 - (-2 + 8)}{2} = \frac{0 - (6)}{2} = \frac{-6}{2} = \boxed{-3}$$

$$\boxed{x = \frac{-2}{\sqrt{3}}}$$

$$-1 \qquad\qquad -1$$
$$-3x^2 = -4$$
$$\frac{-3x^2}{-3} = \frac{-4}{-3}$$
$$x^2 = \frac{4}{3}$$

$$x = \pm\sqrt{\frac{4}{3}}$$
$$x = \pm\frac{\sqrt{4}}{\sqrt{3}}$$
$$x = \pm\frac{2}{\sqrt{3}}$$
Note $\frac{2}{\sqrt{3}} \notin [-2,0]$

Here are three consequences of the Mean Value Theorem:

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**The Constant Function Theorem:**

If $f$ is differentiable on an interval $I$ and the derivative $f'(x) = 0$ for all $x \in I$, then $f$ is constant on $I$.

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**Corollary:**

If $f$ and $g$ are differentiable on an interval $I$ and $f'(x) = g'(x)$ for all $x \in I$, then $f - g$ is constant on $I$; that is, $f(x) = g(x) + c$ where $c$ is a constant.

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**Increasing/ Decreasing Function Theorem:**

If $f$ is differentiable on an interval $I$ and $f'(x) > 0$ for all points $x \in I$, then $f$ is <u>increasing</u> on $I$.

If $f$ is differentiable on an interval $I$ and $f'(x) < 0$ for all points $x \in I$, then $f$ is <u>decreasing</u> on $I$.

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**Proof of Increasing Function Theorem:**

Suppose $f'(x) > 0$ on an interval $I$. We wish to show that $f(a) < f(b)$ for any pair $a < b$ in $I$. Let $a$ and $b$ be any pair of point in $I$ satisfying $a < b$. Then

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some $c$ satisfying $a < c < b$. Now, $f'(c) > 0$ and so

$$\frac{f(b) - f(a)}{b - a} > 0$$

so $f(b) - f(a) > 0$, since $b - a > 0$. Therefore, $f(a) < f(b)$. This argument works for **any** $a < b$ in $I$, therefore $f(x)$ is increasing on the whole interval.

The Decreasing Function Theorem and Constant Function Theorem can be proved with similar arguments.

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**Example 13:** Let $f(x) = \dfrac{x+4}{x+7}$. Find the intervals over which the function is increasing.

$$f'(x) = \frac{(x+7)(1) - (x+4)(1)}{(x+7)^2} = \frac{x + 7 - x - 4}{(x+7)^2} = \frac{3}{(x+7)^2} > 0 \quad \text{for all } x \text{ in } \mathbb{R} \text{ except } x = -7$$

Note: $f$ is not defined at $x = -7$

Consequently, $f$ is increasing on $(-\infty, -7) \cup (-7, \infty)$

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**Example 14:** Find the largest interval or collection of intervals on which the function $f(t) = t^4 - 10t^2 + 9$ is increasing.
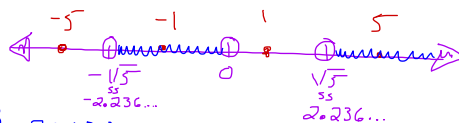
$f'(t) = 4t^3 - 20t$

Goal: Determine when $4t^3 - 20t > 0$

Note $4t^3 - 20t = 0$

$4t(t^2 - 5) = 0$

$4t = 0 \qquad t^2 - 5 = 0$

$t = 0 \qquad\quad t^2 = 5$

$\qquad\qquad\quad t = \pm\sqrt{5}$

$-\sqrt{5} \atop \approx -2.236...$   $0$   $\sqrt{5} \atop \approx 2.236...$

The function $f$ is increasing on $(-\sqrt{5}, 0) \cup (\sqrt{5}, \infty)$

| Test Point | $4t$ | $t^2 - 5$ | Sign of $f'(t)$ | Inc/Dec |
|---|---|---|---|---|
| $-5$ | $-$ | $+$ | $-$ | Decreasing |
| $-1$ | $-$ | $-$ | $+$ | Increasing |
| $1$ | $+$ | $-$ | $-$ | Decreasing |
| $5$ | $+$ | $+$ | $+$ | Increasing |

67

**Example 15:** Find the largest value of $A$ such that the function $h(s) = \dfrac{1}{(s-9)^4}$ is increasing for all $s$ in the interval $(-\infty, A)$.

$h(s) = (s-9)^{-4}$

$h'(s) = -4(s-9)^{-3} \cdot \dfrac{d}{ds}(s-9)$

$h'(s) = -4(s-9)^{-3} \cdot 1$

$h'(s) = \dfrac{-4}{(s-9)^3}$ ← Never Zero undefined when $s-9=0$

So Critical point when $s-9=0$

$s-9=0$
$\phantom{s}+9 \ +9$
$s=9$

| Test Value | $-4$ | $(s-9)^3$ | Sign of $h'(s)$ | Inc/Dec |
|---|---|---|---|---|
| ⓪ | $-$ | $-$ | $+$ | Increasing |
| 10 | $-$ | $+$ | $-$ | Decreasing |

So $h(s)$ is increasing on $(-\infty, 9)$ which means $A=9$

**First derivative test for (local) maxima and minima:** If $f$ has a critical ~~value~~ point at $x=c$, then

- $f$ has a local maximum at $x=c$ if the sign of $f'$ around $c$ is $\quad \dfrac{+\ +\ +\ -\ -\ -\ -\ -}{c}$

- $f$ has a local minimum at $x=c$ if the sign of $f'$ around $c$ is $\quad \dfrac{-\ -\ -\ -\ -\ +\ +\ +}{c}$
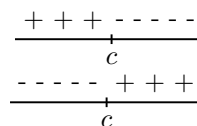
**Example 16:** Suppose $g(t) = 6(t-5)^3 - 131$. Find the critical numbers of $g(t)$. Then determine which critical ~~numbers~~ points give rise to local maxima and which critical numbers give rise to local minima.

$g'(t) = 6\cdot 3(t-5)^2$ ← Defined everywhere
$\phantom{g'(t)} = 18(t-5)^2$

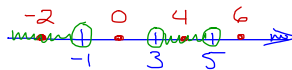So Critical point when
$t-5=0$
$\phantom{t}+5 \ +5$
$t=5$

There is no change in direction, since the function is always increasing. Consequently, $g(t)$ has no local maxima or local minima.

| Test Value | 18 | $(t-5)^2$ | Sign of $g'(t)$ | Inc/Dec |
|---|---|---|---|---|
| ⓪ | $+$ | $+$ | $+$ | Increasing |
| 6 | $+$ | $+$ | $+$ | Increasing |

**Example 17:** Suppose $k'(t) = (t-5)(t+1)(t-3)$. Find all intervals on which the function $k(t)$ is decreasing.

Defined everywhere

So Critical points when
$(t-5)(t+1)(t-3) = 0$

$t-5=0$ or $t+1=0$ or $t-3=0$
$\phantom{t}+5 \ +5 \qquad -1 \ -1 \qquad +3 \ +3$
$t=5 \qquad\quad t=-1 \qquad\quad t=3$

| Test Value | $t-5$ | $t+1$ | $t-3$ | Sign of $k'(t)$ | Inc/Dec |
|---|---|---|---|---|---|
| $-2$ | $-$ | $-$ | $-$ | $-$ | Decreasing |
| 0 | $-$ | $+$ | $-$ | $+$ | Increasing |
| 4 | $-$ | $+$ | $+$ | $-$ | Decreasing |
| 6 | $+$ | $+$ | $+$ | $+$ | Increasing |

∪ min
∧ max
∪ min

$K(t)$ is decreasing on $(-\infty, -1) \cup (3, 5)$

68

**Example 18:** Suppose $\dfrac{du}{dx} = (x^2+1)(x-3)(x-1)(x+5)$. Find the $x$ value in the interval $[-5,3]$ where $u(x)$ takes its maximum value.

Defined everywhere
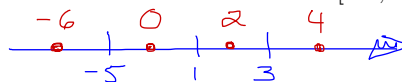
So Critical points when

$(x^2+1)(x-3)(x-1)(x+5)=0$

$x^2+1=0 \quad x-3=0 \quad x-1=0 \quad x+5=0$
$\quad -1\ -1 \quad\ +3\ +3 \quad +1\ +1 \quad -5\ -5\ -5$
$x^2=-1 \quad\ x=3 \quad\ x=1 \quad\ x=-5$

Impossible

So $u(x)$ take it's maximum value at $x=1$ on the interval $[-5,3]$ and this maximum value is $u(1)$

| Test Value | $x^2+1$ | $x-3$ | $x-1$ | $x+5$ | Sign of $u'(x)$ | Inc/Dec |
|---|---|---|---|---|---|---|
| $-6$ | $+$ | $-$ | $-$ | $-$ | $-$ | Decreasing |
| $0$ | $+$ | $-$ | $-$ | $+$ | $+$ | Increasing |
| $2$ | $+$ | $-$ | $+$ | $+$ | $-$ | Decreasing |
| $4$ | $+$ | $+$ | $+$ | $+$ | $+$ | Increasing |

**Example 19:** Suppose $g'(x) = 1 + x^2 + x^4$. Find the $x$ values in the interval $[-3,4]$ where $g(x)$ takes its minimum.

Defined everywhere

So Critical points when

$1+x^2+x^4=0$ ← Impossible

Let $u=x^2$
then $u^2=(x^2)^2=x^4$

$1+u+u^2=0$
$u^2+u+1=0$

Discriminant $=b^2-4ac$
$= 1^2-4(1)(1)$
$= 1-4 = -3 < 0$ NO Real Solutions

So No critical points

In fact Note $g'(x) = 1+x^2+x^4 = 1+$ NON-Negative # $> 0$
Consequently, the function is always increasing, which means in the interval $[-3,4]$ $g(x)$ takes its minimum at $x=-3$ and the minimum is $g(-3)$.

**Example 20:** Find the $t$ value(s) in the interval $(0,\infty)$ where $s(t)$ takes its minimum, given that

$s(t) = \dfrac{e^{t/2}}{t}$.

$s'(t) = \dfrac{t\frac{d}{dt}(e^{t/2}) - e^{t/2}\frac{d}{dt}(t)}{t^2}$

$= \dfrac{t\cdot e^{t/2}\cdot\frac{1}{2} - e^{t/2}\cdot 1}{t^2}$

$= \dfrac{e^{t/2}\cdot[\frac{1}{2}t-1]}{t^2}$

undefined at $t=0$
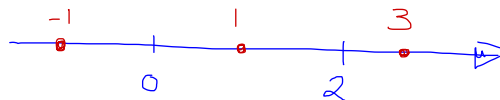Zero when $e^{t/2}\cdot[\frac{1}{2}t-1]=0$

$e^{t/2}=0$ or $\frac{1}{2}t-1=0$
Impossible! $\qquad\quad +1\ +1$
$\qquad\qquad\qquad \frac{1}{2}t=1$
$\qquad\qquad\qquad 2[\frac{1}{2}t]=2[1]$
$\qquad\qquad\qquad t=2$

Consequently, the critical points are $t=0$ and $t=2$

| Test Points | $e^{t/2}$ | $\frac{1}{2}t-1$ | $t^2$ | Sign of $s'(t)$ | Inc/Dec |
|---|---|---|---|---|---|
| $-1$ | $+$ | $-$ | $+$ | $-$ | Decreasing |
| $1$ | $+$ | $-$ | $+$ | $-$ | Decreasing |
| $3$ | $+$ | $+$ | $+$ | $+$ | Increasing |

So $s(t)$ has a minimum at $x=2$ on the interval $(0,\infty)$ and the minimum is

$s(2) = \dfrac{e^{2/2}}{2} = \boxed{\dfrac{e}{2}}$

69

► **Curve sketching:** Information on the first derivative can be used to help us sketch the graph of a function. For example, the first derivative can be used to determine where a function is increasing and where it is decreasing.

**Example 21:** Find the intervals where the function $f(x) = x^3 - 3x^2 + 1$ is increasing and the ones where it is decreasing. Use this information to sketch the graph of $f(x) = x^3 - 3x^2 + 1$. ← *y-intercept = (0,1)*

*$f'(x) = 3x^2 - 6x$ (Defined everywhere)*    *Local max @ x=0 of 1*    *$f(2) = 2^3 - 3(2)^2 + 1 = 8 - 12 + 1 = -3$*

*So Critical points when*    *Local min @ x=2 of -3*

*$3x^2 - 6x = 0$*

*$3x(x-2) = 0$*

*$3x = 0$ or $x - 2 = 0$*

*$\frac{3x}{3} = \frac{0}{3}$     +2 +2*

*$x = 0$     $x = 2$*

| Test Value | $3x$ | $x-2$ | Sign of $f'(x)$ | Inc/Dec |
|---|---|---|---|---|
| $-1$ | $-$ | $-$ | $+$ | Increasing |
| $1$ | $+$ | $-$ | $-$ | Decreasing |
| $3$ | $+$ | $+$ | $+$ | Increasing |
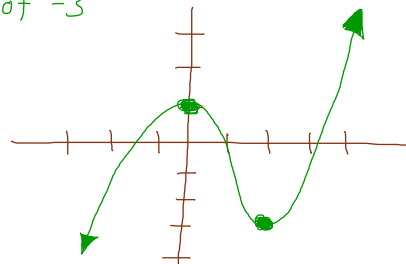


► **Concavity:** We saw that the first derivative of a function gives us information about where the function is increasing and decreasing. What graphical information, if any, can be obtained from the second derivative? We begin with an informal discussion. Lets suppose $y = f(x)$ is twice differentiable on the interval $[a, b]$ and further suppose $f''(x) > 0$ on the entire interval. Since the derivative of $f'(x)$ is positve on $[a, b]$, then $f'(x)$ is increasing on $[a, b]$. Now, $f'(x)$ measures the steepness of the the graph of $y = f(x)$. Since $f'(x)$ is increasing, then $y = f(x)$ is gets steeper as $x$ increases. Finally, increasing steepness of the graph should result in an "upward bending" tendency of the graph.

**Definition of Concavity:**

More formally, ~~we say~~ *one says* $y = f(x)$ is **concave up** on an interval $I$ if, for any pair of points $a < b$ in $I$, the secant line through $(a, f(a))$ and $(b, f(b))$ lies above the graph of $y = f(x)$ for $a < x < b$.

We say $y = f(x)$ is **concave down** on an interval $I$ if, for any pair of points $a < b$ in $I$, the secant line through $(a, f(a))$ and $(b, f(b))$ lies below the graph of $y = f(x)$ for $a < x < b$.

Using the above definition to verify that a function is concave up or concave down can be rather difficult. The next two theorems provide easier methods for checking if a function is concave up or concave down.

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**Tangent line characterization of concavity:** Suppose $f(x)$ is differentiable on an interval $[a, b]$.

- The graph of $y = f(x)$ is concave up on $[a, b]$ if and only if, for each $a < x_0 < b$, the graph of $y = f(x)$ lies above the tangent line at $x = x_0$.

- The graph of $y = f(x)$ is concave down on $[a, b]$ if and only if, for each $a < x_0 < b$, the graph of $y = f(x)$ lies below the tangent line at $x = x_0$.

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**Second derivative test for concavity:** Suppose $f(x)$ is *twice* differentiable on an interval $[a, b]$.
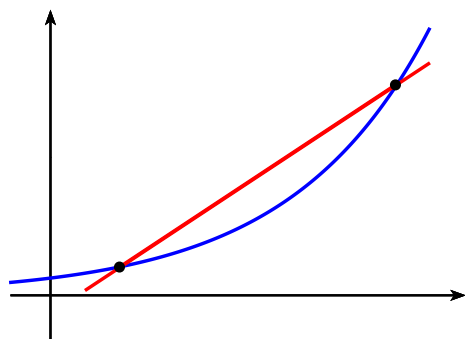
- The graph of $y = f(x)$ is concave up on $[a, b]$ if and only if $f''(x) \geq 0$ for all $x$ in $[a, b]$.

- The graph of $y = f(x)$ is concave down on $[a, b]$ if and only if $f''(x) \leq 0$ for all $x$ in $[a, b]$.
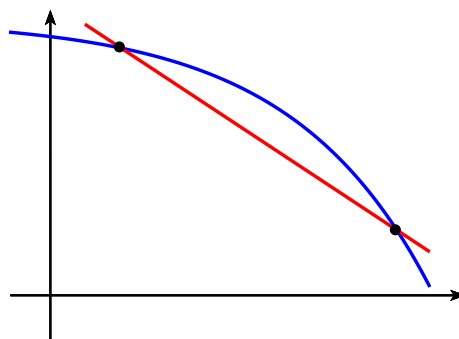
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70
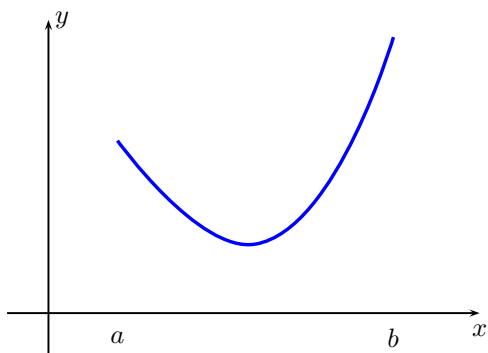
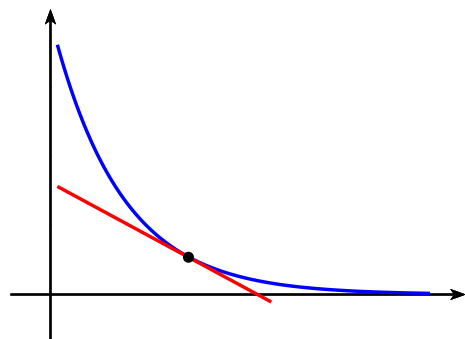graph of function concave upward on $[a, b]$
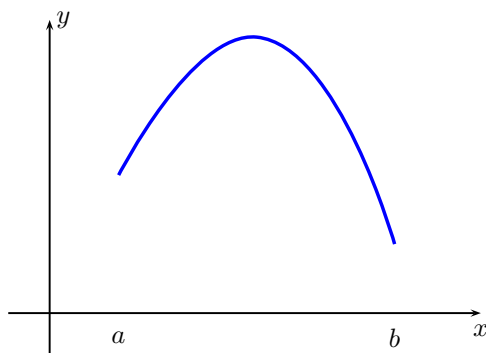

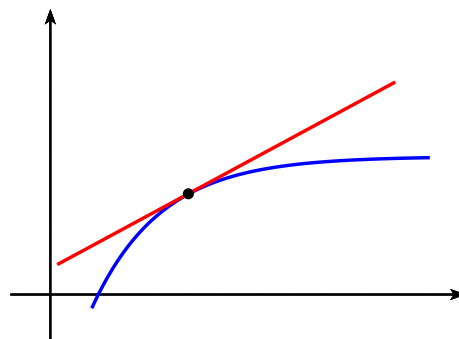graph of function concave downward on $[a, b]$
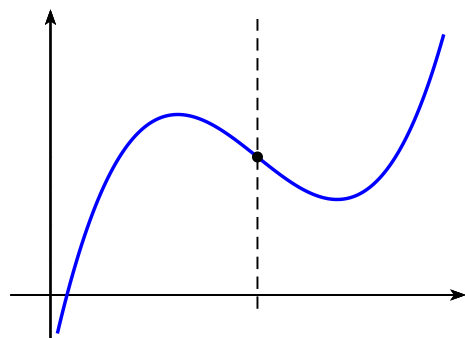

secant line lies above concave up graph
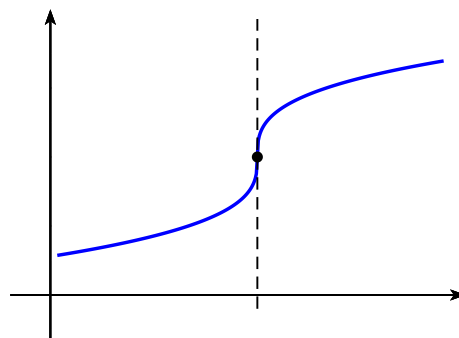

secant line lies below concave down graph


tangent line lies below concave up graph


tangent line lies above concave down graph


graph with varying concavity


graph with varying concavity

71

**Example 22:** Find the intervals over which the function $f(x) = x^4 - 6x^3 + 12x^2 + 3x - 1$ is concave upward and the ones over which it is concave downward.

$f'(x) = 4x^3 - 18x^2 + 24x + 3$

$f''(x) = 12x^2 - 36x + 24$  Defined everywhere

$= 12(x^2 - 3x + 2)$

$= 12(x-2)(x-1)$

$X - 2 = 0$ or $X - 1 = 0$
    $+2 \ +2$       $+1 \ +1$
    $X = 2$         $X = 1$

Need to know when $f''(x) = 0$

$12(x-2)(x-1) = 0$  Divide by 12

$(x-2)(x-1) = 0$

| Test Value | 12 | $(x-2)$ | $(x-1)$ | Sign of $f''(x)$ | Concave |
|---|---|---|---|---|---|
| 0 | + | − | − | + | upward |
| 1.5 | + | − | + | − | downward |
| 3 | + | + | + | + | upward |

Conclusion:

$f(x)$ is concave upward on $(-\infty, 1) \cup (2, \infty)$

$f(x)$ is concave downward on $(1,2)$

**Inflection points:** A point $(c, f(c))$ on the graph is called a **point of inflection** if the graph of $y = f(x)$ changes concavity at $x = c$. That is, if the graph goes from concave up to concave down, or from concave down to concave up. If $(c, f(c))$ is a point of inflection on the graph of $y = f(x)$ and if the second derivative is defined at this point, then $f''(c) = 0$.

Thus, points of inflection on the graph of $y = f(x)$ are found where either $f''(x) = 0$ or the second derivative is not defined. However, if either $f''(x) = 0$ or the second derivative is not defined at a point, it is not necessarily the case that the point is a point of inflection. Care must be taken.

**Example 23:** If the derivative of the function $g(x)$ is given by $g'(x) = 4x^2 + 12x + 15$, determine the interval(s) where $g(x)$ is concave upward and the one(s) where it is concave downward. Find the $x$-coordinate of the inflection point(s).
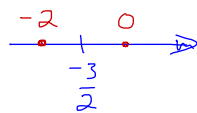
$g''(x) = 8x + 12$  Defined everywhere

$= 4(2x + 3)$

Potential points of inflection when $-\frac{3}{2}$

$4(2x+3) = 0$  Divide by 4

$2x + 3 = 0$
$\quad -3 \ -3$
$\frac{2x}{2} = \frac{-3}{2}$
$X = -\frac{3}{2}$
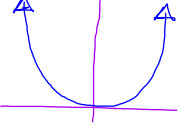
| Test Value | 4 | $2x+3$ | Sign of $g''(x)$ | Concave |
|---|---|---|---|---|
| −2 | + | − | − | downward |
| 0 | + | + | + | upward |

change

Conclusion:

$g(x)$ is concave downward on $(-\infty, -\frac{3}{2})$

$g(x)$ is concave upward on $(-\frac{3}{2}, \infty)$

$g(x)$ has a point of inflection at $x = \frac{-3}{2}$ since the concavity changes at $x = -\frac{3}{2}$

**Example 24:** Suppose $f(x) = x^4$. Determine the interval(s) where $f(x)$ is concave upward and the one(s) where it is concave downward. Find the $x$-coordinate of the inflection point(s).

Note:

$f(x) = x^4$
always concave upward
No points of inflection.

$f'(x) = 4x^3$

$f''(x) = 12x^2 \geq 0$  so $f(x)$ is concave upward on $\mathbb{R}$
which means there is no change in concavity
so no points of inflection.

Note: $f''(0) = 12(0)^2 = 0$ but $x = 0$ is not a point of inflection.

72

**Example 25:** Find the $x$-coordinate of the inflection points of the function $g(x) = e^{-x^2}$.

$g'(x) = -2x e^{-x^2}$

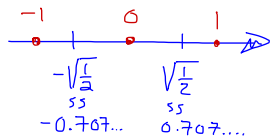$g''(x) = \frac{d}{dx}(-2x) \cdot e^{-x^2} + (-2x) \cdot \frac{d}{dx}(e^{-x^2})$

$= -2e^{-x^2} - 2x \cdot (-2x)e^{-x^2}$

$= -2e^{-x^2} + 4x^2 e^{-x^2}$

$= 2e^{-x^2}[-1 + 2x^2]$

Need to know when $g''(x) = 0$

$2e^{-x^2}(-1 + 2x^2) = 0$

$2e^{-x^2} = 0$ or $-1 + 2x^2 = 0$

Impossible
$e^{power} \neq 0$

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm\sqrt{\frac{1}{2}}$

Number line: $-1$, $0$, $1$; $-\sqrt{\frac{1}{2}}$ ≈ $-0.707...$, $\sqrt{\frac{1}{2}}$ ≈ $0.707...$

| Test Value | $2e^{-x^2}$ | $-1+2x^2$ | Sign of $g''(x)$ | Concave | |
|---|---|---|---|---|---|
| $-1$ | $+$ | $+$ | $+$ | upward | Change |
| $0$ | $+$ | $-$ | $-$ | downward | Change |
| $1$ | $+$ | $+$ | $+$ | upward | |

Test values: $-\sqrt{\frac{1}{2}}$, $\sqrt{\frac{1}{2}}$

**Conclusion:** $x = \sqrt{\frac{1}{2}}$ and $x = -\sqrt{\frac{1}{2}}$ are the $x$-coordinate of the points of inflection because the concavity changes at the points.

**Example 26:** Let $h(x) = xe^{-x}$. Find the interval over which $h(x)$ is concave downward.

$h'(x) = \frac{d}{dx}(x) \cdot e^{-x} + x \cdot \frac{d}{dx}(e^{-x})$

$= 1 \cdot e^{-x} + x(-1)e^{-x}$

$= e^{-x} - xe^{-x}$ ← $h'(x)$

$h''(x) = \frac{d}{dx}(e^{-x}) - \frac{d}{dx}(xe^{-x})$

$= -e^{-x} - [e^{-x} - xe^{-x}]$

$= -e^{-x} - e^{-x} + xe^{-x}$

$= -2e^{-x} + xe^{-x}$ ← Defined everywhere
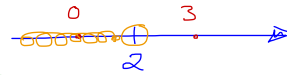
Need to know when $h''(x) = 0$

$-2e^{-x} + xe^{-x} = 0$

$e^{-x}[-2 + x] = 0$

$e^{-x} = 0$ or $-2 + x = 0$

Impossible
$+2 \quad +2$

$x = 2$

Number line: $0$, $3$, with $2$ marked.

| Test Value | $e^{-x}$ | $-2+x$ | sign of $h''(x)$ | Concave |
|---|---|---|---|---|
| $0$ | $+$ | $-$ | $-$ | downward |
| $3$ | $+$ | $+$ | $+$ | upward |

$h(x)$ is concave downward on $(-\infty, 2)$

**Example 27:** Suppose $p(x) = x^3 + ax^2 + bx + c$, for some unknown constants $a$, $b$, and $c$. Suppose $p(x)$ has a local minimum at $p(3) = -8$ and $p(x)$ has an inflection point at $x = 1$. Determine $p(2)$.

(**Hint:** Use the inflection point and local minimum conditions to first determine $a$, $b$, and $c$.)

$p'(x) = 3x^2 + 2ax + b$

$p''(x) = 6x + 2a$

$p'(3) = 0$

however, $p'(3) = 3(3)^2 - 6(3) + b$
$= 27 - 18 + b$
$= 9 + b$

So $0 = 9 + b \longrightarrow b = -9$

So $p(x) = x^3 - 3x^2 - 9x + c$

$p''(1) = 0$

however $p''(1) = 6(1) + 2a = 6 + 2a$

So we have $0 = 6 + 2a$
$\frac{-6 \quad -6}{}$
$\frac{-6}{2} = \frac{2a}{2}$
$a = -3$

Note: $-8 = p(3) = 3^3 - 3(3)^2 - 9(3) + c$
$-8 = 27 - 27 - 27 + c$
$-8 = -27 + c$
$-8 + 27 = c$
$c = 19$

So $p(x) = x^3 - 3x^2 - 9x + 19$

So $p'(x) = 3x^2 - 6x + b$

$p(2) = 2^3 - 3(2)^2 - 9(2) + 19$
$= 8 - 12 - 18 + 19 = \boxed{-3}$

**Example 28:** The graph of the _derivative_ $f'$ of a function $f$ is shown.

(a) On what intervals is $f$ increasing or decreasing?

(b) At what values of $x$ does $f$ have a local maximum or minimum?

(c) On what intervals is $f$ concave upward or downward?

(d) State the $x$-coordinate of the inflection points of $f$.

$f$ is increasing on $(1, 6] \cup (8, \infty)$

$f$ is decreasing on $(-\infty, 1) \cup (6, 8)$

local maximum at $x = 6$
local minimum at $x = 1$ and $x = 8$

$f$ is concave upward on $(-\infty, 2] \cup (3, 5) \cup (7, \infty)$

$f$ is concave downward on $(2, 3) \cup (5, 7)$

Points of inflection at $x = 2, x = 3, x = 5 \{x = 7$

[graph labeled "derivative of $f$" with $y$ and $x$ axes, curve in red; lower graph labeled $f''(x)$]

73