

MA123, Chapter 6: Extreme values, Mean Value Theorem, Curve sketching, and Concavity

Chapter Goals:

- Apply the Extreme Value Theorem to find the global extrema for continuous function on closed and bounded interval.
- Understand the connection between critical points and local extreme values.
- Understand the relationship between the sign of the derivative and the intervals on which a function is increasing and on which it is decreasing.
- Understand the statement and consequences of the Mean Value Theorem.
- Understand how the derivative can help you sketch the graph of a function.
- Understand how to use the derivative to find the global extreme values (if any) of a continuous function over an unbounded interval.
- Understand the connection between the sign of the second derivative of a function and the concavities of the graph of the function.
- Understand the meaning of inflection points and how to locate them.

Assignments:

- |               |               |
|---------------|---------------|
| Assignment 12 | Assignment 13 |
| Assignment 14 | Assignment 15 |

Finding the largest profit, or the smallest possible cost, or the shortest possible time for performing a given procedure or task, or figuring out how to perform a task most productively under a given budget and time schedule are some examples of practical real-world applications of Calculus. The basic mathematical question underlying such applied problems is how to find (if they exist) the largest or smallest values of a given function on a given interval. This procedure depends on the nature of the interval.

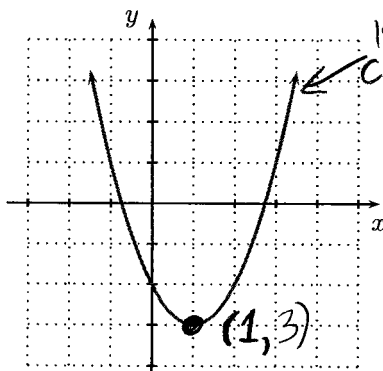
► **Global (or absolute) extreme values:** The largest value a function (possibly) attains on an interval is called its **global (or absolute) maximum value**. The smallest value a function (possibly) attains on an interval is called its **global (or absolute) minimum value**. Both maximum and minimum values (if they exist) are called **global (or absolute) extreme values**.

Example 1(a):

Find the maximum and minimum values for the function

$$f(x) = (x - 1)^2 - 3,$$

if they exist.



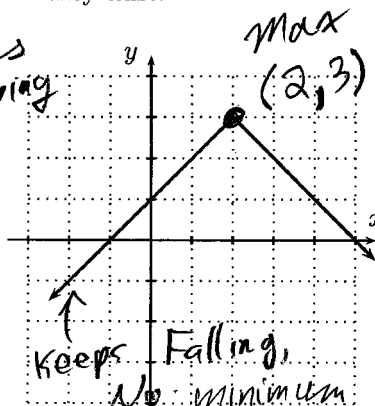
min at  $x=1, y=-3$   
No max

Example 1(b):

Find the maximum and minimum values for the function

$$f(x) = -|x - 2| + 3,$$

if they exist.



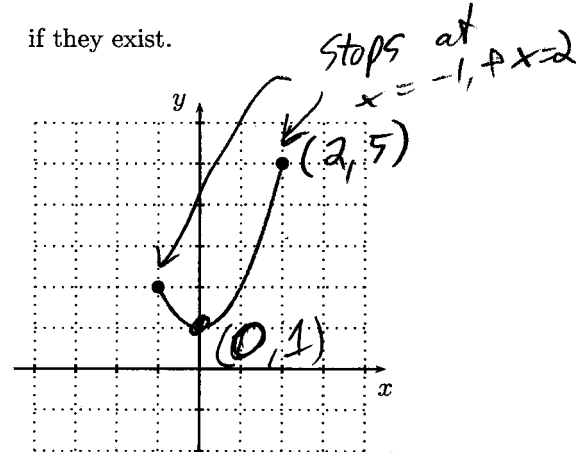
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Example 1(c):

Find the maximum and minimum values for the function

$$f(x) = x^2 + 1, \quad x \in [-1, 2]$$

if they exist.



max at  $x=2, y=5$   
min at  $x=0, y=1$

We first focus on continuous functions on a closed and bounded interval. The question of largest and smallest values of a continuous function  $f$  on an interval that is not closed and bounded requires us to pay more attention to the behavior of the graph of  $f$ , and specifically to where the graph is rising and where it is falling.

**Closed and bounded intervals:**

An interval is **closed and bounded** if it has finite length and contains its endpoints.

For example, the interval  $[-2, 5]$  is closed and bounded.

► **The Extreme Value Theorem (EVT):**

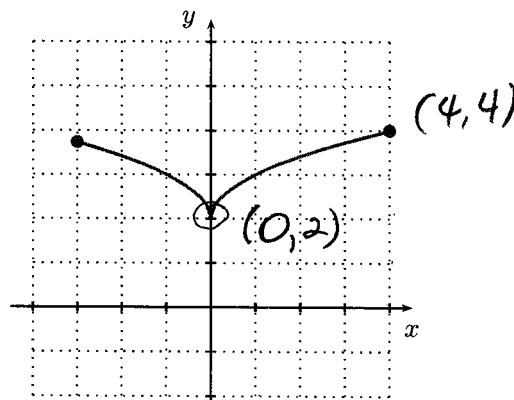
If a function  $f$  is continuous on a closed, bounded interval  $[a, b]$ , then the function  $f$  attains a maximum and a minimum value on  $[a, b]$ .

**Example 2(a):** Let  $f(x) = \begin{cases} 2 + \sqrt{x} & \text{if } x > 0 \\ 2 + \sqrt{-x} & \text{if } x \leq 0 \end{cases}$

Does  $f(x)$  have a maximum and a minimum value on  $[-3, 4]$ ? How does this example illustrate the Extreme Value Theorem?

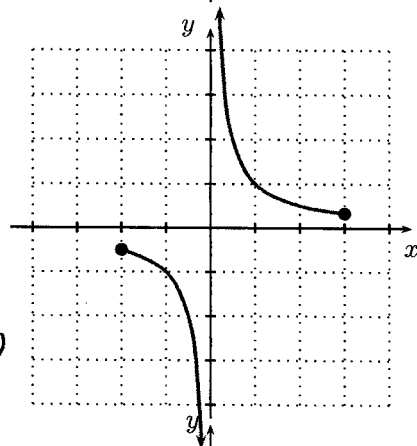
min at  $x=0, y=2$ , Max at  $x=4, y=4$

$f(x)$  is continuous on closed, bounded interval, so  $f(x)$  must have a max & min.



**Example 2(b):** Let  $g(x) = \frac{1}{x}$ . Does  $g(x)$  have a maximum value and a minimum value on  $[-2, 3]$ ? Does this example contradict the Extreme Value Theorem? Why or why not?

$g(x)$  has no max & no min  
This does not contradict EVT, as  $g(x)$  is not continuous, so we cannot use EVT to draw conclusions about  $g(x)$



**Example 2(c):** Let  $h(x) = x^4 - 2x^2 + 1$ . Does  $h(x)$  have a maximum value and a minimum value on  $(-1.25, 1.5)$ ? Does this example contradict the Extreme Value Theorem? Why or why not?

min at  $x = -1$  &  $x = 1$

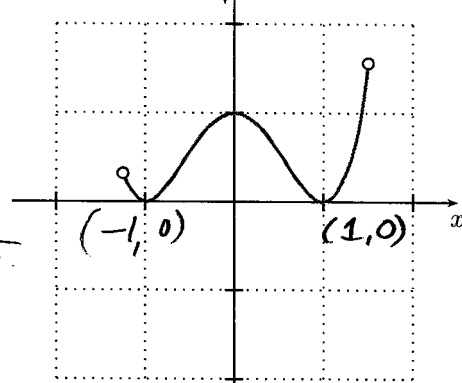
No max

Max appears to be at  $x = 1.5, y = 1.5625$ .

However,  $h(x)$  is only defined for  $x < 1.5$ .

So we can never get to  $x = 1.5$ .

this example shows the necessity of closed interval in the EVT.



The EVT is an existence statement; it doesn't tell you how to locate the maximum and minimum values of  $f$ .

The following results tell you how to narrow down the list of possible points on the given interval where the function  $f$  might have an extreme value to (usually) just a few possibilities. You can then evaluate  $f$  at these few possibilities, and pick out the smallest and largest value.

► **Fermat's Theorem:** Let  $f(x)$  be a continuous function on the interval  $[a, b]$ . If  $f$  has an extreme value at a point  $c$  strictly between  $a$  and  $b$ , and if  $f$  is differentiable at  $x = c$ , then  $f'(c) = 0$ .

► **Corollary:** Let  $f(x)$  be a continuous function on the closed, bounded interval  $[a, b]$ . If  $f$  has an extreme value at  $x = c$  in the interval, then either

- $c = a$  or  $c = b$ ;
- $a < c < b$  and  $f'(c) = 0$ ;
- $a < c < b$  and  $f$  is not differentiable at  $x = c$ , so that  $f'$  is not defined at  $x = c$ .

Note:  $f'(x) = 0 \Rightarrow x = 3$  or  $-1$   
 Why did we ignore  $x = -1$ ?  
 we are only concerned with  $x \in [0, 4]$ , in this example.

**Example 3:** Find the maximum and minimum values of  $f(x) = x^3 - 3x^2 - 9x + 5$  on the interval  $[0, 4]$ .

For which values  $x$  are the maximum and minimum values attained?

Extreme values occur where

- Endpoints,  $x = 0$  or  $4$ .
- Where  $f'(x) = 0$
- Where  $f'(x)$  DNE

$$f'(x) = 3x^2 - 6x - 9 = 3(x-3)(x+1) = 0 \Rightarrow x = 3, x = -1$$

Also,  $f'(x)$  exists everywhere

$$f(0) = 0^3 - 3 \cdot 0^2 - 9 \cdot 0 + 5 = 5$$

$$f(3) = 3^3 - 3 \cdot 3^2 - 9 \cdot 3 + 5 = -22$$

$$f(4) = 4^3 - 3 \cdot 4^2 - 9 \cdot 4 + 5 = -15$$

Min at  $x = 3$ , min value =  $-22$   
 Max at  $x = 0$ , Max value =  $5$

**Example 4:** Find the maximum and minimum values of  $F(s) = \frac{2s+1}{s-6}$  on the interval  $[-1, 5]$ . For which values  $s$  are the maximum and minimum values attained?

$F(s)$  is not defined at  $s = 6$ , but  $s = 6$  is outside of  $[-1, 5]$ .

Max & Min occur at

- $s = -1$  or  $s = 5$
- Where  $F'(s) = 0$
- Where  $F'(s) = \text{DNE}$

Now, Quotient Rule  $\Rightarrow$

$$F'(s) = \frac{2 \cdot (s-6) - (2s+1)}{(s-6)^2}$$

$$F'(s) = \frac{-13}{(s-6)^2}$$

Now,  $F'(s) = \text{DNE}$  at  $s = 6$ , but  $s = 6$  is outside of  $[-1, 5]$ .

$$F'(s) = 0 \Rightarrow \frac{-13}{(s-6)^2} = 0$$

$$\Rightarrow -13 = 0 \cdot (s-6)^2$$

$$\Rightarrow -13 = 0 \Rightarrow F'(s) \text{ never zero}$$

Only need to check endpoints.

$$F(-1) = \frac{2(-1)+1}{-1-6} = \frac{-1}{-7} = \frac{1}{7}$$

$$F(5) = \frac{2 \cdot 5 + 1}{5-6} = \frac{11}{-1} = -11$$

Max at  $s = -1$ , Max =  $\frac{1}{7}$   
 Min at  $s = 5$ , Min =  $-11$

**Example 5:** Find the maximum and minimum values of  $f(x) = x^{2/3}$  on the interval  $[-1, 8]$ . For which values  $s$  are the maximum and minimum values attained?

Max & Min could occur

- At  $x = -1$  or  $x = 8$
- Where  $f'(x) = 0$
- Where  $f'(x) = \text{DNE}$

Now

$$f'(x) = \frac{2}{3} x^{-1/3} = \frac{2}{3 \sqrt[3]{x}}$$

Now

$$f'(x) = 0 \Rightarrow \frac{2}{3 \sqrt[3]{x}} = 0$$

$$\Rightarrow 2 = 0, \text{ so } f'(x) \text{ never } = 0.$$

Also,  $f'(x) = \frac{2}{3 \sqrt[3]{x}}$  is not defined at  $x = 0$  (0 in denominator)

So, need to check  $x = -1, 0, 8$

$$f(-1) = (-1)^{2/3} = ((-1)^2)^{1/3} = 1$$

$$f(0) = 0^{2/3} = 0$$

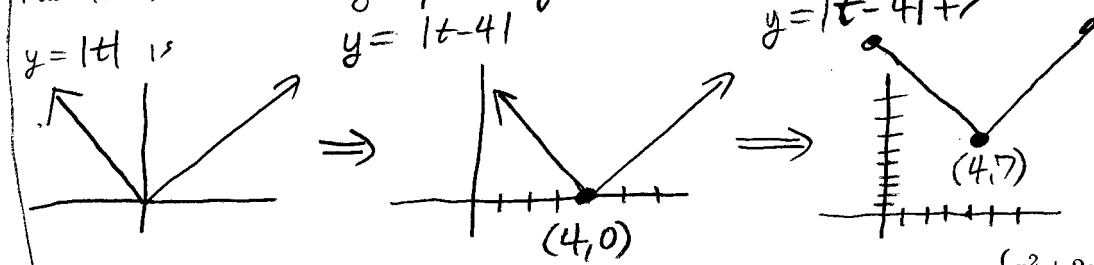
$$f(8) = 8^{2/3} = (8^{1/3})^2 = 2^2 = 4$$

Max at  $x = 8$ , Max =  $4$   
 Min at  $x = 0$ , min =  $0$

In order to apply Fermat's method, need  $k(x)$  to be continuous on  $[-2, 3]$ . Only possible trouble spot is  $x=1$   
 But  $\lim_{x \rightarrow 1^-} x^2 + 2x + 1 = 4 = \lim_{x \rightarrow 1^+} 3x + 7$ , so  $k(x)$  is continuous on  $[-2, 3]$ .

**Example 6:** Find the  $t$  values on the interval  $[-10, 10]$  where  $g(t) = |t-4| + 7$  takes its maximum and minimum values. What are the maximum and minimum values?

This is best done graphically:



min at  $t=4, y=7$   
 Max at either  $t=-10, 10$   
 $g(10) = |10-4| + 7 = 13$   
 $g(-10) = |-10-4| + 7 = 21$   
 Max at  $t=-10, y=21$

**Example 7:** Find the maximum and minimum values of  $k(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \leq 1 \\ -3x + 7 & \text{if } x > 1 \end{cases}$  on the interval  $[-2, 3]$ .

$$k'(x) = \begin{cases} 2x + 2 & x < 1 \\ -3 & x > 1 \end{cases}$$

$k'(x)$  not defined at  $x=1$   
 $k'(x) = 0 \Rightarrow 2x + 2 = 0 \Rightarrow x = -1$

Max/min at  $x = -2, -1, 1, \text{ or } 3$ .  
 $k(-2) = (-2)^2 + 2(-2) + 1 = 1$   
 $k(-1) = (-1)^2 + 2(-1) + 1 = 0$   
 $k(1) = 1^2 + 2 \cdot 1 + 1 = 4$   
 $k(3) = -3 \cdot 3 + 7 = -2$   
 Max at  $x=1, y=4$   
 Min at  $x=3, y=-2$

**Example 8:** Find the maximum and minimum values of  $g(x) = 1 + x + x^2 + x^3$  on the interval  $[0, 2]$ . For which values  $x$  are the maximum and minimum values attained?

$$g'(x) = 1 + 2x + 3x^2$$

$g'(x)$  exists everywhere.  
 $g'(x) = 0 \Rightarrow 1 + 2x + 3x^2 = 0$   
 But  $x$  in  $[0, 2]$ , in particular

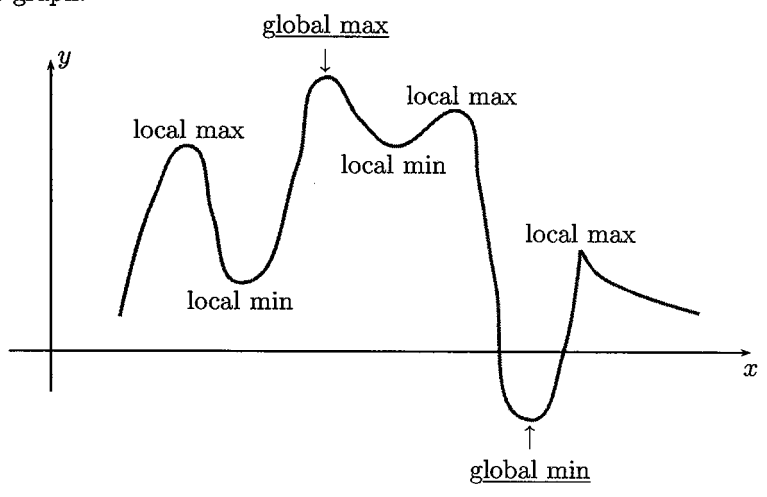
$x$  is positive, so  $1 + 2x + 3x^2 \geq 1 + 0 + 0 = 1$   
 so  $g'(x) \neq 0$  for  $x$  in  $[0, 2]$ .  
 Extreme values only at  $x=0$  or  $x=2$ .

$$g(0) = 1 + 0 + 0^2 + 0^3 = 1$$

$$g(2) = 1 + 2 + 2^2 + 2^3 = 15$$

Max at  $x=0, y=1$   
 Min at  $x=2, y=15$

**Local (or relative) extreme points:** In addition to the points where a function might have a maximum or minimum value, there are other points that are important for the behavior of the function and the shape of its graph.



If you think of the graph of the function as the profile of a landscape, the global maximum could represent the highest hill in the landscape, while the minimum could represent the deepest valley. The other points indicated in the graph, which look like tops of hills (although not the highest hills) and bottom of valleys (although not the deepest valleys), are called local (or relative) extreme values.

**Definition:** A function  $f$  has a **local (or relative) maximum** at a point  $(c, f(c))$  if there is some interval about  $c$  such that  $f(c) \geq f(x)$  for all  $x$  in that interval. A function  $f$  has a **local (or relative) minimum** at a point  $(c, f(c))$  if there is some interval about  $c$  such that  $f(c) \leq f(x)$  for all  $x$  in that interval.

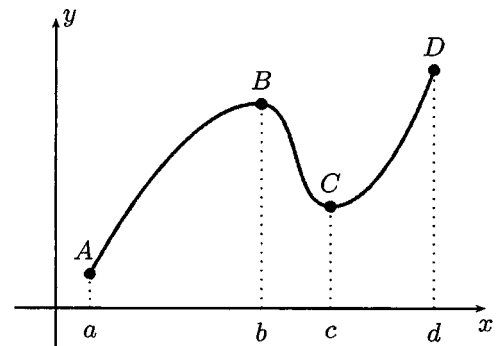
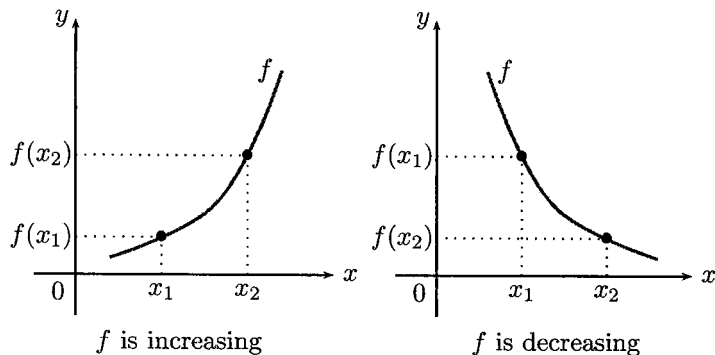
**Theorem:** If  $f$  has a local extreme value at  $(c, f(c))$  and is differentiable at that point  $c$ , then  $f'(c) = 0$

**Critical points:** Let  $f$  be a function. If  $f$  is defined at the point  $x = c$  and either  $f'(c) = 0$  or  $f'(c)$  is undefined then the point  $c$  is called a **critical point** of  $f$ .

► **Increasing and decreasing functions:** A function  $f$  is said to be increasing when its graph rises and decreasing when its graph falls. More precisely, we say that:

$f$  is **increasing** on an interval  $I$  if  $f(x_1) < f(x_2)$  whenever  $x_1 < x_2$  in  $I$ .

$f$  is **decreasing** on an interval  $I$  if  $f(x_1) > f(x_2)$  whenever  $x_1 < x_2$  in  $I$ .



$f$  is increasing on  $[a, b]$  and  $[c, d]$   
 $f$  is decreasing on  $[b, c]$ .

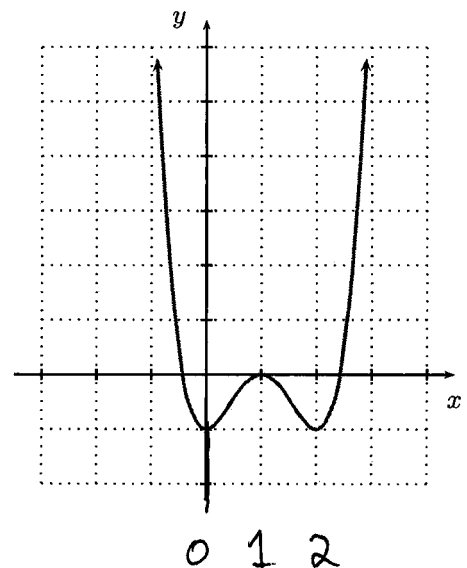
**Example 9:** The picture shows the graph of  $y = f(x)$ .

(a) Find the intervals on which  $f(x)$  is increasing and decreasing.

Increasing on  $(0, 1) \cup (2, \infty)$   
 Decreasing on  $(-\infty, 0) \cup (1, 2)$

(b) Find the intervals on which the tangent line to  $f(x)$  has positive slope, and the intervals on which the tangent line has negative slope.

Positive slope on  $(0, 1) \cup (2, \infty)$   
 Negative slope on  $(-\infty, 0) \cup (1, 2)$



(a, b) Possible:  $f(x) \nearrow$  on  $(-\infty, -1.5)$ , so just need  $f(-2) < f(-1.5)$   
 (c) No:  $f(x) \searrow$  on  $(-1.5, 2)$ , so need  $f(-1.5) > f(-1) > f(2)$   
 $12 > 0 > 1$  No  
 (d) No: Need  $f(-1.5) > f(0) > f(2)$   
 $12 > 0 > 1$

**Example 10:** Suppose that  $f(-1.5) = 12$  and  $f(2) = 1$ . In addition, you are given that  $f(x)$  is increasing on the intervals  $(-\infty, -1.5) \cup (2, \infty)$  and  $f(x)$  is decreasing on the interval  $(-1.5, 2)$ . Which of the following are not possible?

(e)  $f(x) \nearrow$  on  $(2, \infty)$  so need  $f(2) < f(3)$   
 $1 < -1$  NO!

(a)  $f(-2) = 7$  Yes  
 (b)  $f(-2) = 11$  Yes  
 (c)  $f(-1) = 13$  No  
 (d)  $f(0) = 0$  No

(f) Need  $f(2) < f(4)$   
 $1 < 6$  Yes

(e)  $f(3) = -1$  No  
 (f)  $f(4) = 6$  Yes  
 (g)  $f(3) = 5$  and  $f(4) = 4$  No  
 (h)  $f(3) = 5$  and  $f(4) = 6$  Yes

(g) Need  $f(2) < f(3) < f(4)$   
 $1 < 5 < 4$  No  
 (h) Need  $f(2) < f(3) < f(4)$   
 $1 < 5 < 6$  Yes

In general, using the algebraic definition of increasing/decreasing to verify that a function is increasing and decreasing can be difficult. However, example 9 suggests an easier method for checking where a function is increasing or decreasing:

- If  $f(x)$  is increasing then  $f'(x) > 0$ ;
- If  $f(x)$  is decreasing then  $f'(x) < 0$ .

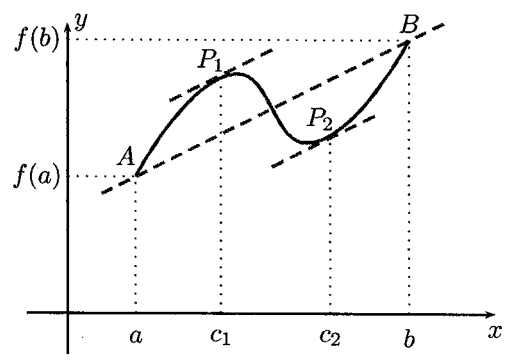
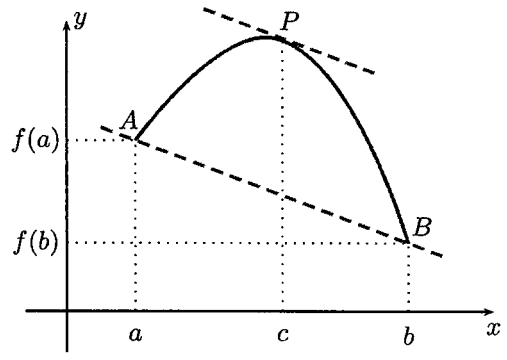
We would like to be able to reverse these implications, but doing so will require some additional theory.

► **The Mean Value Theorem (MVT):**

If  $f$  is continuous on  $[a, b]$  and differentiable at every point strictly between  $a$  and  $b$ , then there exists some point  $x = c$  (and maybe more than one) strictly between  $a$  and  $b$  such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

**Geometric interpretation of the MVT:** For some (non-necessarily unique) point  $c$  between  $a$  and  $b$  the tangent line to the graph of  $f$  at  $P(c, f(c))$  has the same slope as the secant line connecting the points  $A(a, f(a))$  and  $B(b, f(b))$  on the graph of  $f$ .



**A rewording of the MVT:**

If  $f$  is continuous on  $[a, b]$  and differentiable at every point strictly between  $a$  and  $b$ , then there exists some point  $x = c$  (and maybe more than one) strictly between  $a$  and  $b$  such that

$$\text{the average rate of change of } f \text{ on } [a, b] = \text{the instantaneous rate of change of } f \text{ at } x = c.$$

**Example 11:** Let  $Q(t) = t^2$ . Find a value  $A \neq 1$  such that the average rate of change of  $Q(t)$  from 1 to  $A$  equals the instantaneous rate of change of  $Q(t)$  at  $t = 2$ .

Need  $ARC = Inst. ROC$   
 $\frac{Q(A) - Q(1)}{A - 1} = Q'(2)$   
 $\frac{A^2 - 1}{A - 1} = 2 \cdot 2 \Rightarrow \frac{(A-1)(A+1)}{A-1} = 4$   
 $\Rightarrow A + 1 = 4$   
 $\Rightarrow A = 3$

$f(x)$  is continuous & differentiable for all  $x$ .

**Example 12:** Let  $f(x) = x - x^3$ . Verify that the function satisfies the hypotheses of the Mean Value Theorem on the interval  $[-2, 0]$ . Then find all numbers  $c$  that satisfy the conclusion of the Mean Value Theorem.

$$\text{AVOC on } [-2, 0] = \frac{f(0) - f(-2)}{0 - (-2)}$$

$$= \frac{(0 - 0^3) - (-2 - (-2)^3)}{2} = -3$$

Inst. ROC =  $1 - 3x^2 \rightarrow c^2 = 4/3 \Rightarrow c = \pm \sqrt{4/3} = \pm \frac{2}{\sqrt{3}}$

So  $1 - 3c^2 = -3 \Rightarrow 3c^2 = 4$  But only want  $c$  in  $[-2, 0]$ .

So  $c = -\frac{2}{\sqrt{3}}$  only

Here are three consequences of the Mean Value Theorem:

**The Constant Function Theorem:**

If  $f$  is differentiable on an interval  $I$  and the derivative  $f'(x) = 0$  for all  $x \in I$ , then  $f$  is constant on  $I$ .

**Corollary:**

If  $f$  and  $g$  are differentiable on an interval  $I$  and  $f'(x) = g'(x)$  for all  $x \in I$ , then  $f - g$  is constant on  $I$ ; that is,  $f(x) = g(x) + c$  where  $c$  is a constant.

**Increasing/ Decreasing Function Theorem:**

If  $f$  is differentiable on an interval  $I$  and  $f'(x) > 0$  for all points  $x \in I$ , then  $f$  is increasing on  $I$ .

If  $f$  is differentiable on an interval  $I$  and  $f'(x) < 0$  for all points  $x \in I$ , then  $f$  is decreasing on  $I$ .

**Proof of Increasing Function Theorem:**

Suppose  $f'(x) > 0$  on an interval  $I$ . We wish to show that  $f(a) < f(b)$  for any pair  $a < b$  in  $I$ . Let  $a$  and  $b$  be any pair of point in  $I$  satisfying  $a < b$ . Then

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some  $c$  satisfying  $a < c < b$ . Now,  $f'(c) > 0$  and so

$$\frac{f(b) - f(a)}{b - a} > 0$$

so  $f(b) - f(a) > 0$ , since  $b - a > 0$ . Therefore,  $f(a) < f(b)$ . This argument works for any  $a < b$  in  $I$ , therefore  $f(x)$  is increasing on the whole interval.

The Decreasing Function Theorem and Constant Function Theorem can be proved with similar arguments.

**Example 13:** Let  $f(x) = \frac{x+4}{x+7}$ . Find the intervals over which the function is increasing.

First find where  $f'(x) = 0$  or DNE

Quotient Rule

$$f'(x) = \frac{(x+7) - (x+4)}{(x+7)^2} = \frac{3}{(x+7)^2}$$

$\Rightarrow f'(x) = 0 \Rightarrow 3 = 0$ , so  $f'(x)$  is never = 0

$f'(x)$  DNE at  $x = -7$ .  
Now,  $f'(x)$  can only change from + to - as we pass through  $x = -7$

Sign Chart

	-7	
$f'(x)$	+++++	+++++
$f(x)$	↗	↘

So,  $f(x)$  increasing on  $(-\infty, -7) \cup (-7, \infty)$ .

**Example 14:** Find the largest interval or collection of intervals on which the function  $f(t) = t^4 - 10t^2 + 9$  is increasing.

First, find where  $f'(t) = 0$  or DNE

$$f'(t) = 4t^3 - 20t = 4t(t^2 - 5)$$

$f'(t)$  exists everywhere,  
 $f'(t) = 0 \Rightarrow t = 0$  or  $t^2 - 5 = 0$   
 $\Rightarrow t = 0, t = \pm \sqrt{5}$

	$-\sqrt{5}$	0	$\sqrt{5}$	
$f'(t)$	----	++++	----	++++
$f(t)$	↘	↗	↘	↗

$f(t)$  ↗ on  $(-\sqrt{5}, 0) \cup (\sqrt{5}, \infty)$

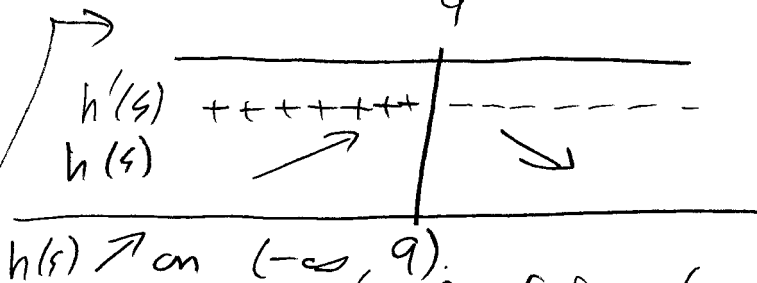
**Example 15:** Find the largest value of  $A$  such that the function  $h(s) = \frac{1}{(s-9)^4}$  is increasing for all  $s$  in the interval  $(-\infty, A)$ .

First,  $h(s)$  not defined at  $s=9$ .

Next,  $h'(s) = \frac{-4}{(s-9)^5}$

$h'(s) = 0 \Rightarrow -4 = 0 \Rightarrow$  Never.

$h'(s)$  DNE when  $s=9$ .



$h(s) \nearrow$  on  $(-\infty, 9)$ .

Largest such interval of form  $(-\infty, A)$  has  $A=9$ .

**First derivative test for (local) maxima and minima:** If  $f$  has a critical value at  $x=c$ , then

- $f$  has a local maximum at  $x=c$  if the sign of  $f'$  around  $c$  is  $\frac{+++-----}{c}$
- $f$  has a local minimum at  $x=c$  if the sign of  $f'$  around  $c$  is  $\frac{-----+++}{c}$

**Example 16:** Suppose  $g(t) = 6(t-5)^3 - 131$ . Find the critical numbers of  $g(t)$ . Then determine which critical numbers give rise to local maxima and which critical numbers give rise to local minima.

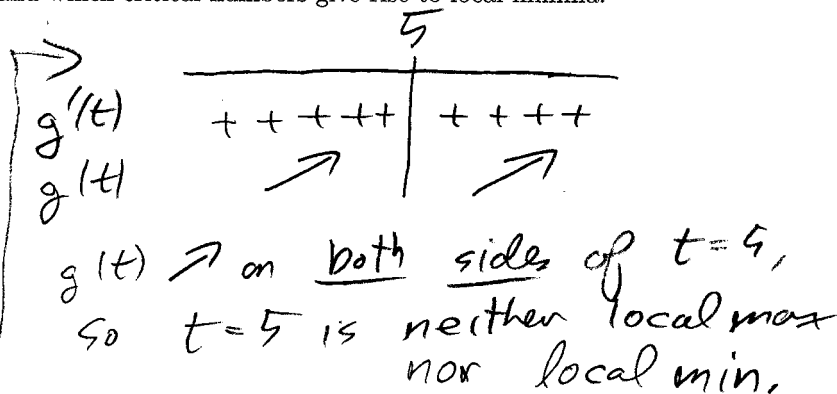
$g'(t) = 6 \cdot 3(t-5)^2 = 18(t-5)^2$

$g'(t)$  exists everywhere.

$g'(t) = 0 \Rightarrow 18(t-5)^2 = 0$

$\Rightarrow t-5=0 \Rightarrow t=5$

only one critical number,  $t=5$



**Example 17:** Suppose  $k(t) = (t-5)(t+1)(t-3)$ . Find all intervals on which the function  $k(t)$  is decreasing.

$k(t) \searrow$  where  $k'(t) < 0$ .

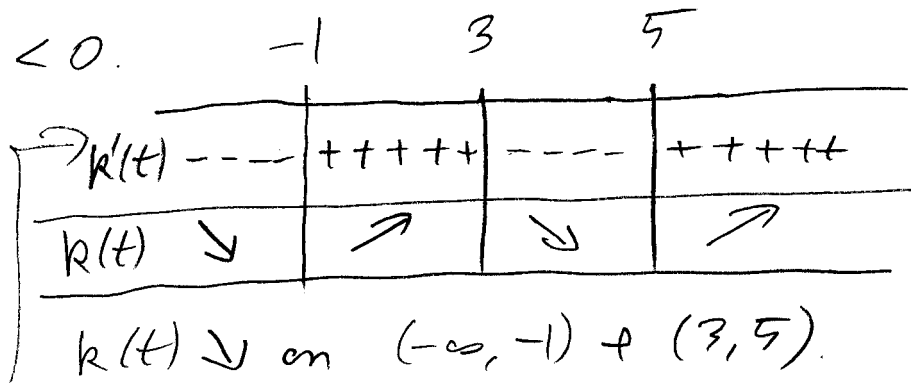
Now

$k'(t) = 0 \Rightarrow$

$(t-5)(t+1)(t-3) = 0$

$\Rightarrow t-5=0$  OR  $t+1=0$   
OR  $t-3=0$

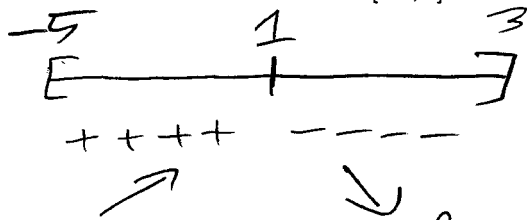
So  $t = -1, 3, 5$





**Example 18:** Suppose  $\frac{du}{dx} = (x^2+1)(x-3)(x-1)(x+5)$ . Find the  $x$  value in the interval  $[-5, 3]$  where  $u(x)$  takes its maximum value.

max at  $-5, 3$ , or where  $u' = 0$ .  
 Now  $\frac{du}{dx} = 0 \Rightarrow (x^2+1)(x-3)(x-1)(x+5) = 0$   
 $\Rightarrow x^2+1=0 \Rightarrow x^2=-1$  NEVER.  
 OR  $x-3=0 \Rightarrow x=3$   
 OR  $x-1=0 \Rightarrow x=1$   
 OR  $x+5=0 \Rightarrow x=-5$   
 max at  $x = -5, 1, \text{ or } 3$ .



$x=1$  gives a local max.  
 As the function only changes from  $\nearrow$  to  $\searrow$  once, we can conclude the local max is actually absolute max.

**Example 19:** Suppose  $g'(x) = 1 + x^2 + x^4$ . Find the  $x$  values in the interval  $[-3, 4]$  where  $g(x)$  takes its minimum.

Min at  $x = -3, 4$  or where  $g'(x) = 0$ , or DNE  
 But  $1+x^2+x^4$  exists everywhere, and  $1+x^2+x^4 = 0$  impossible, since  $1 + \text{positive} = 0$  impossible.  
 In fact,  $g'(x) = 1+x^2+x^4 \geq 1+0+0 \geq 1 > 0$   
 so  $g'(x) > 0$  on  $[-3, 4]$

$\Rightarrow g'(x) > 0 \Rightarrow g(x) \nearrow$  on all of  $[-3, 4]$ ,  
 so minimum occurs at left endpoint,  $x = -3$

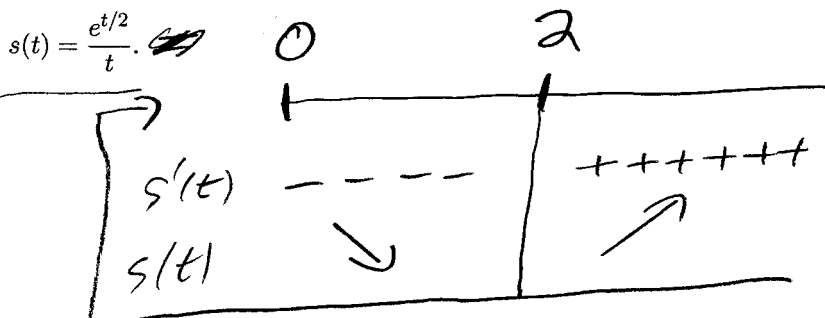
**Example 20:** Find the  $t$  value(s) in the interval  $(0, \infty)$  where  $s(t)$  takes its minimum, given that

Quotient Rule

$$s'(t) = \frac{(t)(e^{t/2})' - (t)'e^{t/2}}{t^2}$$

$$= \frac{t \cdot \frac{1}{2}e^{t/2} - 1 \cdot e^{t/2}}{t^2}$$

$$= \frac{(\frac{1}{2}t - 1)e^{t/2}}{t^2}$$



Minimum at  $t=2$ .

$s'(t) = \text{DNE}$  at  $t=0$   
 $s'(t) = 0 \Rightarrow (\frac{1}{2}t - 1)e^{t/2} = 0$   
 $\Rightarrow \frac{1}{2}t - 1 = 0 \Rightarrow t = 2$

► **Curve sketching:** Information on the first derivative can be used to help us sketch the graph of a function. For example, the first derivative can be used to determine where a function is increasing and where it is decreasing.

**Example 21:** Find the intervals where the function  $f(x) = x^3 - 3x^2 + 1$  is increasing and the ones where it is decreasing. Use this information to sketch the graph of  $f(x) = x^3 - 3x^2 + 1$ .

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

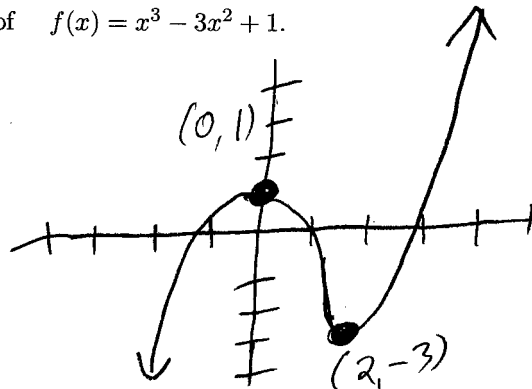
$$f'(x) = 0 \Rightarrow x = 0, x = 2$$

	0	2	
$f'(x)$	+++	---	+++
$f(x)$	↗	↘	↗

Local max at  $x = 0$   
Local min at  $x = 2$

$$f(0) = 0^3 - 3 \cdot 0^2 + 1 = 1$$

$$f(2) = 2^3 - 3 \cdot 2^2 + 1 = -3$$



► **Concavity:** We saw that the first derivative of a function gives us information about where the function is increasing and decreasing. What graphical information, if any, can be obtained from the second derivative? We begin with an informal discussion. Let's suppose  $y = f(x)$  is twice differentiable on the interval  $[a, b]$  and further suppose  $f''(x) > 0$  on the entire interval. Since the derivative of  $f'(x)$  is positive on  $[a, b]$ , then  $f'(x)$  is increasing on  $[a, b]$ . Now,  $f'(x)$  measures the steepness of the the graph of  $y = f(x)$ . Since  $f'(x)$  is increasing, then  $y = f(x)$  is gets steeper as  $x$  increases. Finally, increasing steepness of the graph should result in an "upward bending" tendency of the graph.

**Definition of Concavity:**

More formally, we say  $y = f(x)$  is **concave up** on an interval  $I$  if, for any pair of points  $a < b$  in  $I$ , the secant line through  $(a, f(a))$  and  $(b, f(b))$  lies above the graph of  $y = f(x)$  for  $a < x < b$ .

We say  $y = f(x)$  is **concave down** on an interval  $I$  if, for any pair of points  $a < b$  in  $I$ , the secant line through  $(a, f(a))$  and  $(b, f(b))$  lies below the graph of  $y = f(x)$  for  $a < x < b$ .

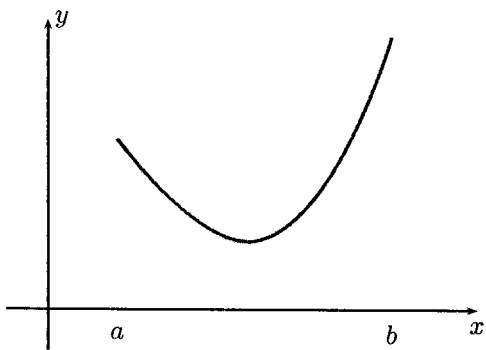
Using the above definition to verify that a function is concave up or concave down can be rather difficult. The next two theorems provide easier methods for checking if a function is concave up or concave down.

**Tangent line characterization of concavity:** Suppose  $f(x)$  is differentiable on an interval  $[a, b]$ .

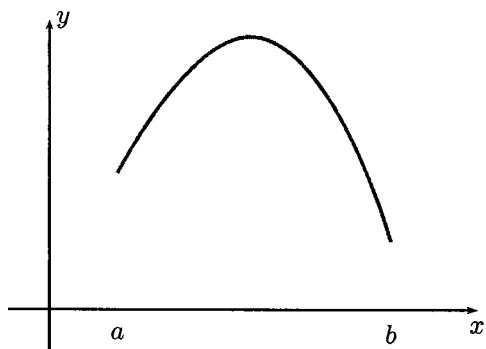
- The graph of  $y = f(x)$  is concave up on  $[a, b]$  if and only if, for each  $a < x_0 < b$ , the graph of  $y = f(x)$  lies above the tangent line at  $x = x_0$ .
- The graph of  $y = f(x)$  is concave down on  $[a, b]$  if and only if, for each  $a < x_0 < b$ , the graph of  $y = f(x)$  lies below the tangent line at  $x = x_0$ .

**Second derivative test for concavity:** Suppose  $f(x)$  is twice differentiable on an interval  $[a, b]$ .

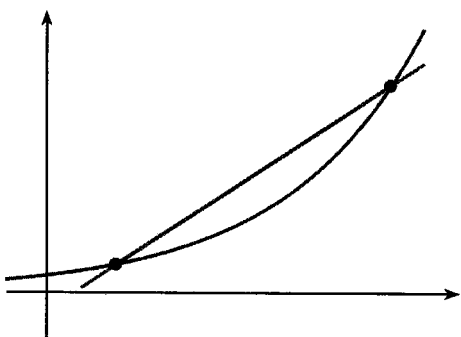
- The graph of  $y = f(x)$  is concave up on  $[a, b]$  if and only if  $f''(x) \geq 0$  for all  $x$  in  $[a, b]$ .
- The graph of  $y = f(x)$  is concave down on  $[a, b]$  if and only if  $f''(x) \leq 0$  for all  $x$  in  $[a, b]$ .



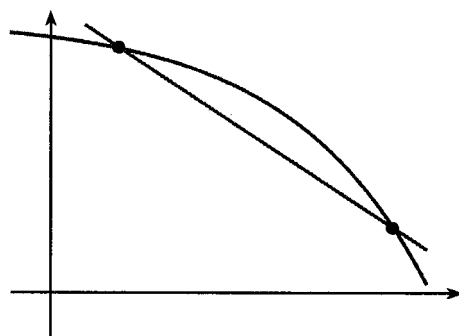
graph of function concave upward on  $[a, b]$



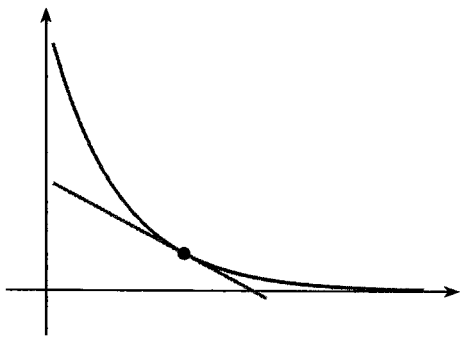
graph of function concave upward on  $[a, b]$



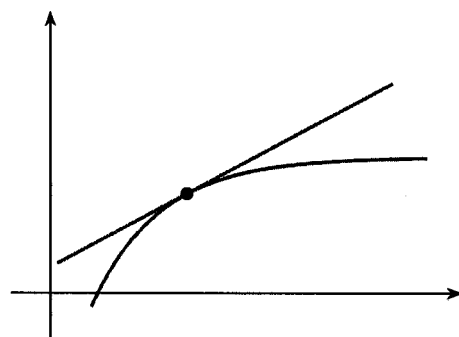
secant line lies above concave up graph



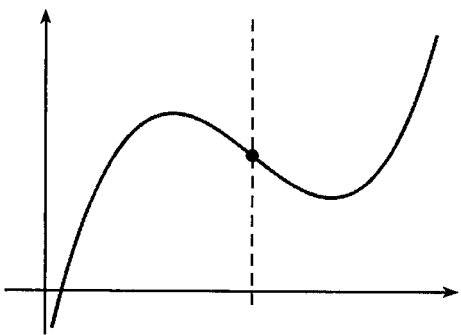
secant line lies below concave down graph



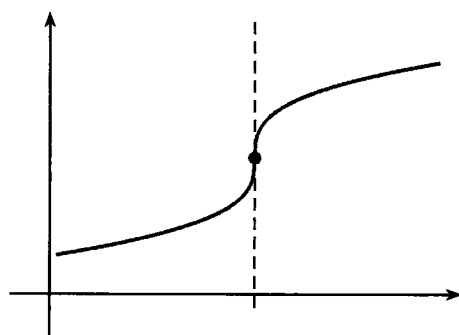
tangent line lies below concave up graph



tangent line lies above concave down graph



graph with varying concavity



graph with varying concavity

**Example 22:** Find the intervals over which the function  $f(x) = x^4 - 6x^3 + 12x^2 + 3x - 1$  is concave upward and the ones over which it is concave downward.

$f'(x) = 4x^3 - 18x^2 + 24x + 3$   
 $f''(x) = 12x^2 - 36x + 24$   
 $= 12(x^2 - 3x + 2)$   
 $= 12(x-2)(x-1)$   
 $f''(x) = 0 \Rightarrow x = 1, 2$

	1	2	
$f''(x)$	++++	----	++++
$f'(x)$	↗	↘	↗
$f(x)$	C. Up	C. Down	C. Up

C. Up on  $(-\infty, 1) \cup (2, \infty)$   
C. Down on  $(1, 2)$

**Inflection points:** A point  $(c, f(c))$  on the graph is called a point of inflection if the graph of  $y = f(x)$  changes concavity at  $x = c$ . That is, if the graph goes from concave up to concave down, or from concave down to concave up. If  $(c, f(c))$  is a point of inflection on the graph of  $y = f(x)$  and if the second derivative is defined at this point, then  $f''(c) = 0$ .

Thus, points of inflection on the graph of  $y = f(x)$  are found where either  $f''(x) = 0$  or the second derivative is not defined. However, if either  $f''(x) = 0$  or the second derivative is not defined at a point, it is not necessarily the case that the point is a point of inflection. Care must be taken.

**Example 23:** If the derivative of the function  $g(x)$  is given by  $g'(x) = 4x^2 + 12x + 15$ , determine the interval(s) where  $g(x)$  is concave upward and the one(s) where it is concave downward. Find the  $x$ -coordinate of the inflection point(s).

$g''(x) = 8x + 12 = 0$   
 $\Rightarrow x = -12/8 = -3/2$

	-3/2	
$g''(x)$	-----	++++
$g'(x)$	↘	↗
$g(x)$	C. Down	C. Up

Concavity switches at  $x = -3/2$ , so  $x = -3/2$  gives rise to an inflection point.

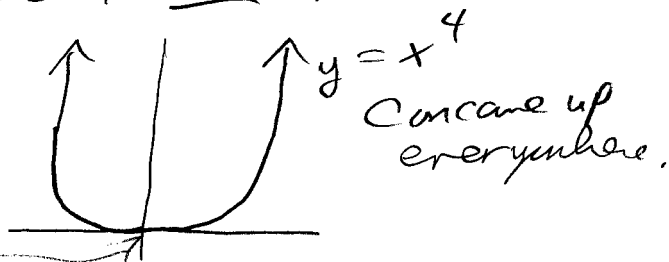
**Example 24:** Suppose  $f(x) = x^4$ . Determine the interval(s) where  $g(x)$  is concave upward and the one(s) where it is concave downward. Find the  $x$ -coordinate of the inflection point(s).

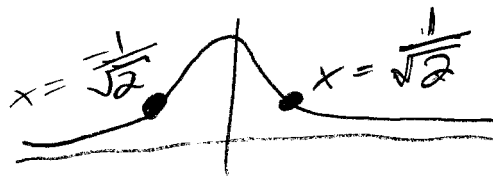
$f'(x) = 4x^3 \Rightarrow f''(x) = 4 \cdot 3 \cdot x^2 = 12x^2$   
 $f''(x) = 0 \Rightarrow 12x^2 = 0 \Rightarrow x = 0$

	0	
$f''(x)$	++++	++++
$f'(x)$	↗	↗
$f(x)$	C. Up	C. Up

$f''(x) = 0$  just meant graph was extremely flat at  $x = 0$ .

Notice,  $f''(0) = 0$ , But concavity does not change, so  $x = 0$  is not inflection pt.





**Example 25:** Find the  $x$ -coordinate of the inflection points of the function  $g(x) = e^{-x^2}$ .

$g'(x) = -2x e^{-x^2}$   
 Chain  
 $g''(x) = (-2x)' e^{-x^2} + (-2x)(e^{-x^2})'$   
 Product  
 $= -2e^{-x^2} + (-2x)(-2x)e^{-x^2}$   
 $= -2(1-2x^2)e^{-x^2}$

$g''(x) = 0 \Rightarrow 1-2x^2 = 0 \Rightarrow x = \pm \frac{1}{\sqrt{2}}$   
 $-1/\sqrt{2}$        $1/\sqrt{2}$

$g''(x)$	++++	-----	++++
$g'(x)$	↗	↘	↗
$g(x)$	C. up	C. Down	C. up

Inflection at  $x = \pm 1/\sqrt{2}$ .

**Example 26:** Let  $h(x) = x e^{-x}$ . Find the interval over which  $h(x)$  is concave downward.

$h'(x) = (x)' e^{-x} + x(e^{-x})'$   
 Product  
 $= e^{-x} + x(-e^{-x})$   
 $= e^{-x}(1-x)$   
 $h''(x) = (1-x)' e^{-x} + (1-x)(e^{-x})'$   
 $= -e^{-x} + (1-x)(-e^{-x})$   
 $= -e^{-x} - (1-x)e^{-x}$

$h''(x) = 0 \Rightarrow (x-2)e^{-x} = 0$   
 $\Rightarrow x = 2$       Concave down on  $(-\infty, 2)$

$h''(x)$	-----	++++
$h'(x)$	↘	↗
$h(x)$	C. Down	C. Up

**Example 27:** Suppose  $p(x) = x^3 + ax^2 + bx + c$ , for some unknown constants  $a, b$ , and  $c$ . Suppose  $p(x)$  has a local minimum at  $p(3) = -8$  and  $p(x)$  has an inflection point at  $x = 1$ . Determine  $p(2)$ .

(Hint: Use the inflection point and local minimum conditions to first determine  $a, b$ , and  $c$ .)

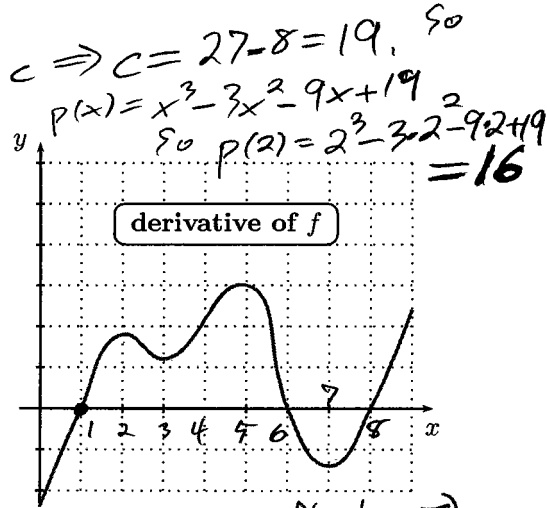
Inflection point at  $x = 1 \Rightarrow p''(1) = 0$   
 $p'(x) = 3x^2 + 2ax + b$   
 $p''(x) = 6x + 2a \Rightarrow p''(1) = 6 + 2a = 0$   
 $\Rightarrow a = -3$   
 $p(x) = x^3 - 3x^2 + bx + c$

Local min at  $x = 3 \Rightarrow p'(3) = 0$   
 $\Rightarrow p'(3) = 3 \cdot 9 - 6 \cdot 3 + b = 0 \Rightarrow b = -9$   
 $\Rightarrow p(x) = x^3 - 3x^2 - 9x + c$

Next,  $p(3) = -8$  so  
 $-8 = 3^3 - 3 \cdot 3^2 - 9 \cdot 3 + c \Rightarrow c = 27 - 8 = 19$   
 $\Rightarrow p(x) = x^3 - 3x^2 - 9x + 19$   
 $\Rightarrow p(2) = 2^3 - 3 \cdot 2^2 - 9 \cdot 2 + 19 = 16$

**Example 28:** The graph of the derivative  $f'$  of a function  $f$  is shown.

- (a) On what intervals is  $f$  increasing or decreasing?
- (b) At what values of  $x$  does  $f$  have a local maximum or minimum?
- (c) On what intervals is  $f$  concave upward or downward?
- (d) State the  $x$ -coordinate of the inflection points of  $f$ .



(a)  $f \uparrow$  where  $f' > 0 \Rightarrow (1, 6), (8, \infty)$   
 $f \downarrow$  where  $f' < 0 \Rightarrow (-\infty, 1), (6, 8)$   
 (b) local max where  $f(x) \uparrow \downarrow$ , so  $x = 6$ ,  
 local min where  $f(x) \downarrow \uparrow$ , so  $x = 1, 8$   
 (c)  $f$  c. up  $\Rightarrow f'' > 0 \Rightarrow f' \uparrow$  so  $(-\infty, 2) \cup (3, 5) \cup (7, \infty)$   
 $f$  c. down  $\Rightarrow f'' < 0 \Rightarrow f' \downarrow$  so  $(2, 3) \cup (5, 7)$

(d) inflection  $\Rightarrow$  concavity changes  
 $\Rightarrow x = 2, 3, 5, 7$

