MA123, Chapter 7a: Related Rates Word Problems (pp. 140-149, Gootman)

Chapter Goals:	In this Chapter we learn a general strategy on how to approach related rates problems, one of the main types of word problems that one usually encounters in a first Calculus course.
Assignments:	Assignment 13

RELATED RATE PROBLEMS

▶ **Overall philosophy and recommended notation:** In a related rate problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). It is almost always better to use Leibniz's notation $\frac{dy}{dt}$, if we are differentiating, for instance, the function y with respect to time t. The y' notation is more ambiguous when working with rates and should therefore be avoided.

▶ **Implicit derivatives:** Imagine you drop a rock in a still pond. This will cause expanding circular ripples in the pond. The area of the outer circle depends on the radius r of the perturbed area:

$$A = \pi r^2.$$

The radius of the outer circle depends on the amount of time t that has elapsed since you dropped the rock. Thus, the area also depends on time. In conclusion, it makes sense to find the rate of change of the area with respect to time and relate it to the rate of change of the radius with to time. We call it an *implicit derivative* as the function A is not explicitly given in terms of t...but only implicitly. We need the chain rule to do this.

▶ **Quick review of the chain rule:** Typically, we are given y as a function of u and u as a function of x, so that we can think of y as a function of x also. The chain rule then says that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example 1: Consider the area of a circle $A = \pi r^2$ and assume that r depends on t. Find a formula for $\frac{dA}{dt}$.

by the chain rule, $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$

we are given $A = \pi r^2$. Using the power rule, we can take the derivative of A with respect to r

$$\frac{dA}{dr} = 2\pi r$$

thus $\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$

▶ **Related rate guideline:** This guideline is found on pp. 143-144 of our textbook.

(1.) Read the problem quickly.

(2.) Read the problem carefully.

(3.) Identify the variables. Note that time is often an understood variable. If the problem involves geometry, draw a picture and label it. Label anything that does not change with a constant. Label anything that does change with a variable.

(4.) Write down which derivatives you are given. Use the units to help you determine which derivatives are given. The word "per" often indicates that you have a derivative.

(5.) Write down the derivative you are asked to find. "How fast..." or "How slowly..." indicates that the derivative is with respect to time.

(6.) Look at the quantities whose derivatives are given and the quantity whose derivative you are asked to find. Find a relationship between all of these quantities.

(7.) Use the chain rule to differentiate the relationship.

(8.) Substitute any particular information the problem gives you about values of quantities at a particular instant and solve the problem. To find all of the values to substitute, you may have to use the relationship you found in step **6**. Take a snapshot of the picture at the particular instant.

Example 2: Boyle's Law states that when a sample gas is compressed at a constant temperature, the pressure P and volume V satisfy the equation $PV = c$, where c is a constant. Suppose that at a certain instant the volume is 600 cm^3, the pressure is 150 kPa, and the pressure is increasing at a rate of 20 kPa/min. At what rate is the volume decreasing at this instant?

We want to find $\frac{dV}{dt}$ for a given time using the formula $PV = c$

Differentiate $PV = c$ with respect to t using product rule.

$$\frac{dP}{dt} \cdot V + P \cdot \frac{dV}{dt} = \frac{dc}{dt}$$

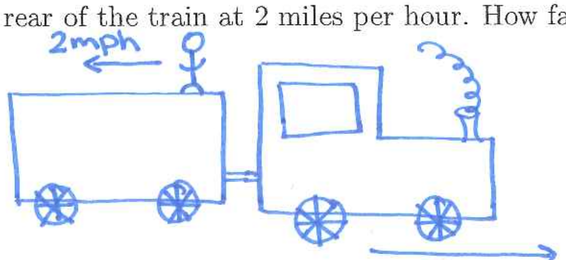
We are given $\frac{dP}{dt} = 20$, $V = 600$, and $P = 150$.
Also, since c is a constant, $\frac{dc}{dt} = 0$

so $$(20)(600) + (150)\frac{dV}{dt} = 0$$

thus $$\frac{dV}{dt} = \frac{-(20)(600)}{150} = \frac{-12000}{150} = \boxed{-80}$$

This means that the volume is decreasing by 80 cm^3/min!

Example 3: A train is traveling over a bridge at 30 miles per hour. A man on the train is walking toward the rear of the train at 2 miles per hour. How fast is the man traveling across the bridge in miles per hour?

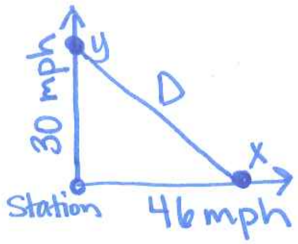


Thus the man is crossing the bridge at a slightly slower rate than the train.

Subtract the velocities!

$30 - 2 = \boxed{28 \text{ mph}}$

Example 4: Two trains leave a station at the same time. One travels north on a track at 30 mph. The second travels east on a track at 46 miles per hour. How fast are they traveling away from one another in miles per hour when the northbound train is 60 miles from the station?



because this is a right triangle, $D^2 = x^2 + y^2$

we want to find $\frac{dD}{dt}$ when $y = 60$

differentiate $D^2 = x^2 + y^2$ with respect to t

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

we are given $\frac{dx}{dt} = 46$, $\frac{dy}{dt} = 30$, and $y = 60$

Since $y = 60$ "$d = rt$" tells us $60 = 30t$ so $t = 2$.

using this, we find x: $x = 46t = 46(2) = 92$

then $D^2 = x^2 + y^2$ gives us $D^2 = (92)^2 + (60)^2 = 12064$

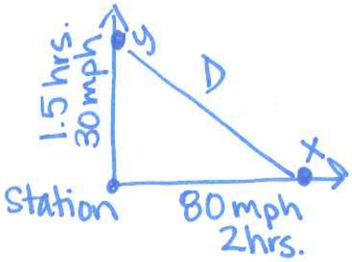
thus $D = \sqrt{12064}$

Now, plugging this into our formula above:

$$2(\sqrt{12064})\frac{dD}{dt} = 2(92)(46) + 2(60)(30)$$

$$\frac{dD}{dt} = \frac{2(92)(46) + 2(60)(30)}{2\sqrt{12064}} = \frac{12064}{2\sqrt{12064}} \approx \boxed{54.92 \text{ mph}}$$

Example 5: Two trains leave a station at 12:00 noon. One travels north on a track at 30 mph. The second travels east on a track at 80 miles per hour. At 1:00 PM the northbound train stops for one-half hour at a station while the eastbound train continues at 80 miles per hour without stopping. At 1:30 PM the northbound train continues north at 30 mph. How fast are the trains traveling away from one another at 2:00 PM?



again, we'll use $D^2 = x^2 + y^2$ so differentiating

still results in $2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

We're given $\frac{dy}{dt} = 30$ and $\frac{dx}{dt} = 80$

$d = r \cdot t$

we can find x & y by $y = (1.5)(30) = 45$ and $x = (2)(80) = 160$

and $D^2 = x^2 + y^2$ gives

$D^2 = (160)^2 + (45)^2$

$D^2 = 27625$

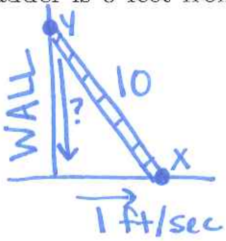
$D = \sqrt{27625}$

using the formula above,

$$2(\sqrt{27625})\frac{dD}{dt} = 2(160)(80) + 2(45)(30)$$

$$\frac{dD}{dt} = \frac{2(160)(80) + 2(45)(30)}{2\sqrt{27625}} = \frac{28300}{2\sqrt{27625}} \approx \boxed{85.13 \text{ mph}}$$

Example 6: A ladder 10 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 feet/sec, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 feet from the wall?



By Pythagorean Theorem, $x^2+y^2=10^2$

differentiate with respect to t! $2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$

We want to find $\frac{dy}{dt}$

We were given $x=6$ and $\frac{dx}{dt}=1$

if $x=6$ and $x^2+y^2=10^2$

$$6^2+y^2=10^2$$

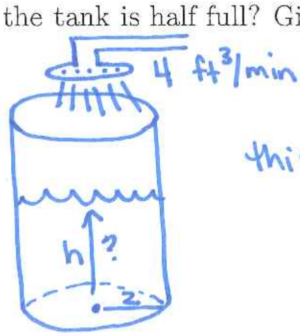
$$y^2=64 \Rightarrow y=8$$

plug in what we know and solve for $\frac{dy}{dt}$

$$2(6)(1)+2(8)\frac{dy}{dt}=0$$

$$\frac{dy}{dt}=\frac{-2(6)(1)}{2(8)}=\frac{-3}{4}=\boxed{-.75\text{ ft/sec}}$$

Example 7: A cylindrical water tank with its circular base parallel to the ground is being filled at the rate of 4 cubic feet per minute. The radius of the tank is 2 feet. How fast is the level of the water in the tank rising when the tank is half full? Give your answer in feet per minute.



The volume of a cylinder is $V=\pi r^2 h$ and in this problem $r=2$, thus $V=\pi(2^2)h$

$$V=4\pi h$$

differentiate! $\frac{dV}{dh}=4\pi$

but wait! We want to find $\frac{dh}{dt}$ so

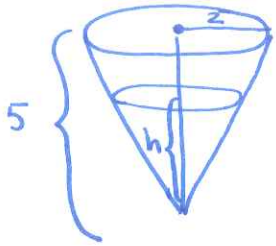
$$\boxed{\frac{dV}{dt}=\frac{dV}{dh}\cdot\frac{dh}{dt}}$$

We are given $\frac{dV}{dt}=4\text{ ft}^3/\text{min}$, plug this in

$$4=4\pi\cdot\frac{dh}{dt}$$

$$\frac{dh}{dt}=\frac{4}{4\pi}=\frac{1}{\pi}\approx\boxed{.32\text{ ft/min}}$$

Example 8: A conical salt spreader is spreading salt at a rate of 3 cubic feet per minute. The diameter of the base of the cone is 4 feet and the height of the cone is 5 feet. How fast is the height of the salt in the spreader decreasing when the height of the salt in the spreader (measured from the vertex of the cone upward) is 3 feet? Give your answer in feet per minute. (It will be a positive number since we use the word "decreasing".)



The volume of a cone is $V = \frac{1}{3}\pi r^2 h$

but r and h are always changing

use similar triangles to find a relationship between r & h

$\frac{r}{h} = \frac{2}{5}$ thus $r = \frac{2}{5}h$

then $V = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h = \frac{1}{3}\pi \cdot \frac{4}{25}h^2 \cdot h = \frac{4}{75}\pi h^3$

differentiate! $\frac{dV}{dt} = 3\left(\frac{4}{75}\pi\right)h^2 \frac{dh}{dt}$

We are given $h=3$ and $\frac{dV}{dt} = -3$ ft³/min (because the volume is decreasing)

thus $-3 = 3\left(\frac{4}{75}\pi\right)(3)^2\frac{dh}{dt}$ so $\frac{dh}{dt} = \frac{-3}{27\left(\frac{4}{75}\pi\right)} = \frac{-25}{12\pi}$

It's decreasing at $\frac{25}{12\pi}$ ft/min.

Example 9: It is estimated that the annual advertising revenue received by a certain newspaper will be

$$R(x) = 0.5x^2 + 3x + 160$$

thousand dollars when its circulation is x thousand. The circulation of the paper is currently 10,000 and is increasing at a rate of 2,000 papers per year. At what rate will the annual advertising revenue be increasing with respect to time 2 years from now?

We want to find $\frac{dR}{dt}$ when $t=2$

differentiate R with respect to t using chain rule

$$\frac{dR}{dt} = \frac{dR}{dx}\cdot\frac{dx}{dt} = \left(2(.5)x+3\right)\frac{dx}{dt} = (x+3)\frac{dx}{dt}$$

an increase of 2,000 papers per year and x is in 1000's gives that $\frac{dx}{dt} = 2$

We need x: Currently at 10,000 papers but increasing by 2,000 per year, so in 2 years $x = 10+2(2) = 14$

then $\frac{dR}{dt} = (x+3)\frac{dx}{dt} = (14+3)(2) = 17\cdot 2 = 34$

Thus revenue will increase $\boxed{\$34{,}000 \text{ per year}}$ when $t=2$.

Example 10: A stock is increasing in value at a rate of 10 dollars per share per year. An investor is buying shares of the stock at a rate of 26 shares per year. How fast is the value of the investor's stock growing when the stock price is 50 dollars per share and the investor owns 100 shares? (**Hint:** Write down an expression for the total value of the stock owned by the investor.

if V = total value of a stock, n = # of shares owned, and P = price per share then $V = n \cdot p$

We want to find $\frac{dV}{dt}$ when $p=50$ and $n=100$

differentiate using the product rule

$$\frac{dV}{dt} = \frac{dn}{dt} \cdot p + n \cdot \frac{dp}{dt}$$

we are given $\frac{dp}{dt} = 10$ and $\frac{dn}{dt} = 26$ plug in with n & p

$$\frac{dV}{dt} = (26)(50) + (100)(10)$$

$$= 2300$$

$$\boxed{\$2300 \text{ per year}}$$

Example 11: Suppose that the demand function q for a certain product is given by

$$q = 4,000\, e^{-0.01 \cdot p},$$

where p denotes the price of the product. If the item is currently selling for \$100 per unit, and the quantity supplied is decreasing at a rate of 80 units per week, find the rate at which the price of the product is changing.

We want $\frac{dp}{dt}$ when $p=100$ and $\frac{dq}{dt} = -80$

differentiate with respect to t.

$$\frac{dq}{dt} = 4000\, e^{-.01p} (-.01) \frac{dp}{dt}$$

$$\frac{dq}{dt} = -40\, e^{-.01p} \frac{dp}{dt}$$

Plug in $p = 100$ and $\frac{dq}{dt} = -80$

$$-80 = -40\, e^{-.01(100)} \frac{dp}{dt}$$

$$\frac{dp}{dt} = \frac{-80}{-40 e^{-1}} = 2e = 5.44$$

$$\boxed{\$5.44 \text{ per week}}$$