

**Chapter Goals:**

- Understand the relationship between the area under a curve and the definite integral.
- Understand the relationship between velocity (speed), distance and the definite integral.
- Use the definite integral to compute the average value of a function over an interval

**Assignments:**

Assignment 18

Assignment 19

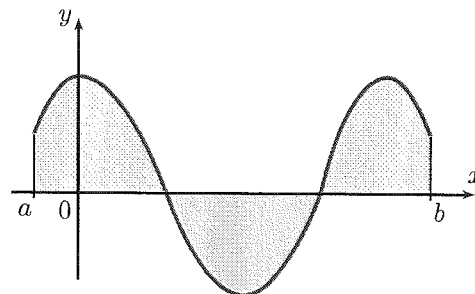
► **The basic idea:** Definite integrals compute *signed area*.

**Definition:** The definite integral

$$\int_a^b f(x) dx$$

computes the signed area between the graph of  $y = f(x)$  and the  $x$ -axis on the interval  $[a, b]$ .

- If  $a < b$  and the region is above the  $x$ -axis, the area has positive sign.
- If  $a < b$  and the region is below the  $x$ -axis, the area has negative sign.
- If the function takes on both positive and negative values on  $[a, b]$ , the “positive” and “negative” areas will cancel out.



That is, if  $a < b$ , then

$$\int_a^b f(x) dx = [\text{area of the region(s) lying above the } x\text{-axis}] - [\text{area of the region(s) lying below the } x\text{-axis}]$$

**Notation:** Given  $\int_a^b f(x) dx$ , we call  $f(x)$  the **integrand**,  $dx$  identifies  $x$  as the variable, and  $a$  and  $b$  are called the **limits of integration**.

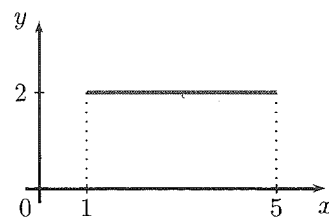
**Applications:** Suppose  $v(t)$  measures the velocity of an object at time  $t$ .

- (a)  $\int_a^b v(t) dt$  measures the displacement of the object from  $t = a$  to  $t = b$ . The displacement is the difference between the object's ending point and starting point.
- (b)  $\int_a^b |v(t)| dt$  measures the total distance traveled between  $t = a$  and  $t = b$ .

If  $v(t)$  is always positive, *displacement* and *distance traveled* are the same.

**Example 1 (Easy area problem):**

Find the area of the region in the  $xy$ -plane bounded above by the graph of the function  $f(x) = 2$ , below by the  $x$ -axis, on the left by the line  $x = 1$ , and on the right by the line  $x = 5$ .



$$\text{Area} = \text{base} \cdot \text{height} = 4(2) = 8$$

**Example 2 (Easy distance traveled problem):**

Suppose a car is traveling due east at a constant velocity of 55 miles per hour. How far does the car travel between noon and 2:00 pm?

$$\text{Distance} = \text{rate} \cdot \text{time} = 55 \frac{\text{mi}}{\text{hr}} \cdot 2 \text{ hr} = 110 \text{ miles}$$

**Example 3:**

Use the graph of  $f(x)$  shown to find the following integrals, given that the shaded region has area A. (Area below  $x$ -axis counts as negative)

(a)  $\int_2^5 f(x) dx = \text{area of triangle}$

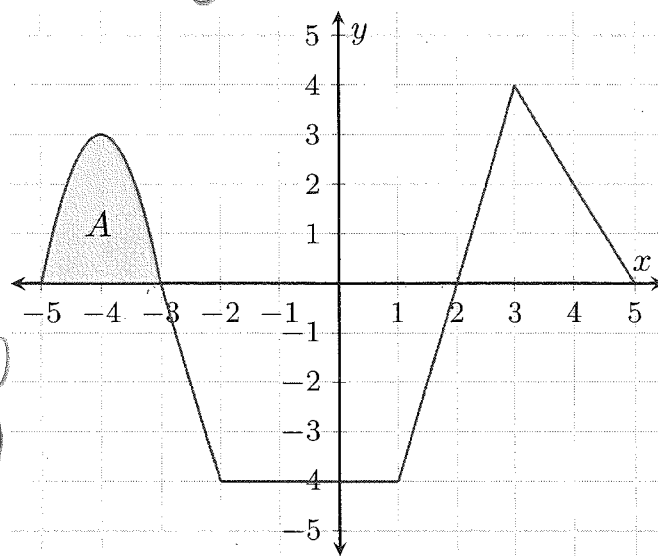
$$= \frac{1}{2}bh = \frac{1}{2}(3)(4)$$

$$= \frac{12}{2} = \boxed{6}$$

(b)  $\int_{-3}^2 f(x) dx = -(\text{Area of trapezoid})$

$$= -\left(h \cdot \frac{b_1 + b_2}{2}\right) = -\left(4 \cdot \frac{5+3}{2}\right)$$

$$= -\left(4 \cdot \frac{8}{2}\right) = \boxed{-16}$$

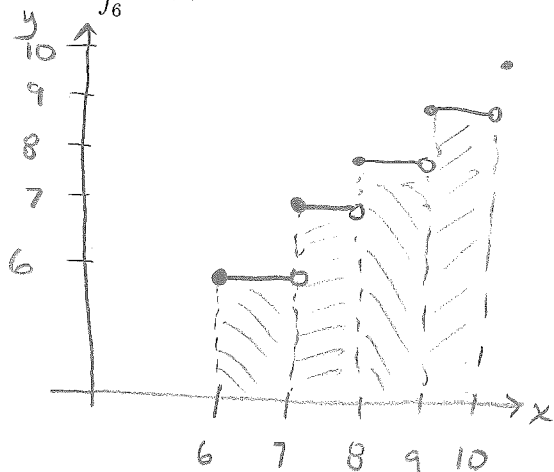


(c)  $\int_{-3}^5 f(x) dx = \underbrace{\int_{-3}^2 f(x) dx}_{\text{trapezoid below } x\text{-axis}} + \underbrace{\int_2^5 f(x) dx}_{\text{triangle above } x\text{-axis}} = -16 + 6 = \boxed{-10}$

(d)  $\int_{-5}^5 f(x) dx = \underbrace{\int_{-5}^{-3} f(x) dx}_{\text{shaded region}} + \underbrace{\int_{-3}^5 f(x) dx}_{\text{answer to (c)}} = \boxed{A - 10}$

**Example 4:** Suppose  $f(x)$  is the greatest integer function, i.e.,  $f(x)$  equals the greatest integer less than or equal to  $x$ . So for example  $f(2.3) = 2$ ,  $f(4) = 4$ , and  $f(6.9) = 6$ .

Find  $\int_6^{10} f(x) dx$ .

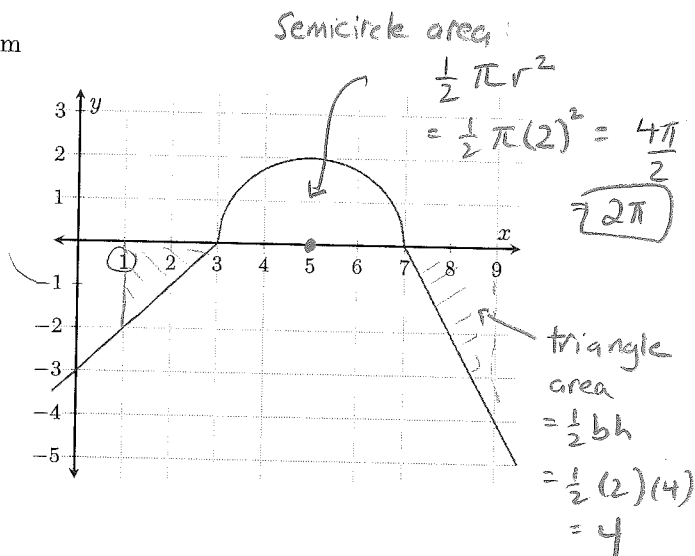


$$\begin{aligned} \int_6^{10} f(x) dx &= \text{area between } x\text{-axis} \\ &\quad \text{and graph of } f(x) \\ &= \text{area of 4 rectangles} \\ &= 6(1) + 7(1) + 8(1) + 9(1) \\ &= 6 + 7 + 8 + 9 \\ &= \boxed{30} \end{aligned}$$

**Example 5:** Consider  $g(x)$  shown here. The graph from  $x = 3$  to  $x = 7$  is a semicircle.

Find  $\int_1^9 g(x) dx$ .

triangle  
area =  
 $\frac{1}{2}bh$   
 $= \frac{1}{2}(2)(2)$   
 $= 2$



$$= \int_1^3 g(x) dx + \int_3^7 g(x) dx + \int_7^9 g(x) dx$$

$$= -2 + 2\pi - 4$$

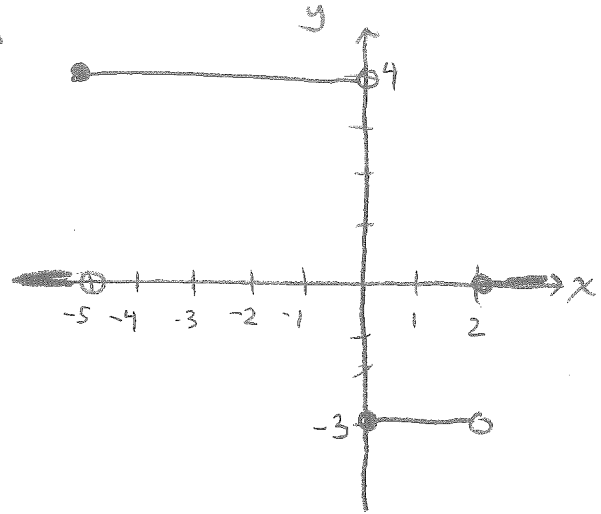
triangle                  semicircle                  triangle  
below axis                  above x-axis                  below x-axis

$$= \boxed{2\pi - 6}$$

**Example 6:** Let

$$y = f(x) \rightarrow$$

$$f(x) = \begin{cases} 0 & \text{if } x < -5 \\ 4 & \text{if } -5 \leq x < 0 \\ -3 & \text{if } 0 \leq x < 2 \\ 0 & \text{if } x \geq 2 \end{cases}$$



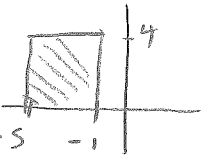
and  $g(x) = \int_{-5}^x f(t) dt$ .

Determine the value of each of the following:

(a)  $g(-10) = \int_{-5}^{-10} f(t) dt = \int_{-5}^{-10} 0 dt = \boxed{0}$

for  $-10 \leq x \leq -5$ ,  $f(x) = 0$  always

(b)  $g(-1) = \int_{-5}^{-1} f(t) dt = \int_{-5}^{-1} 4 dt = 4(4) = \boxed{16}$   
base · height

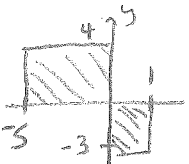


← area of rectangle

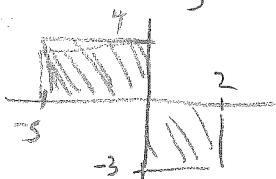
rectangle above x-axis

rectangle below x-axis

(c)  $g(1) = \int_{-5}^1 f(t) dt = \int_{-5}^0 f(t) dt + \int_0^1 f(t) dt$   
 $= 5(4) - 1(3) = 20 - 3 = \boxed{17}$



(d)  $g(6) = \int_{-5}^6 f(t) dt = \int_{-5}^0 f(t) dt + \int_0^2 f(t) dt + \int_2^6 f(t) dt$   
 $= 5(4) - 2(3) + 0$   
 $= 20 - 6 + 0 = \boxed{14}$



(e) What is the absolute maximum of  $g(x)$ ?

→ Largest possible area under  $f(t)$  is  $\boxed{20}$

► **Some properties of definite integrals:**

1.  $\int_a^a f(x) dx = 0$

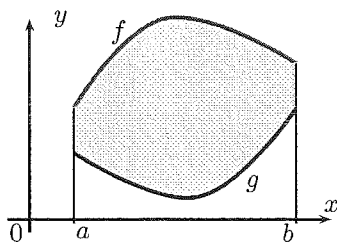
2.  $\int_a^b k f(x) dx = k \int_a^b f(x) dx$

3.  $\int_a^b (f(x) \pm g(x)) dx = \left( \int_a^b f(x) dx \right) \pm \left( \int_a^b g(x) dx \right)$

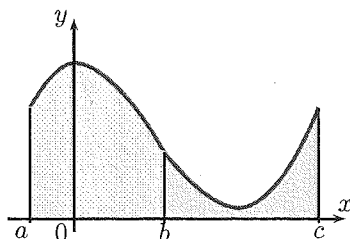
4.  $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$

5.  $\int_a^b f(x) dx = - \int_b^a f(x) dx$

**Geometric illustration of some of the above properties:**



Property 3. says that if  $f$  and  $g$  are positive-valued functions with  $f$  greater than  $g$ , then  $\int_a^b (f(x) - g(x)) dx$  gives the area between the graphs of  $f$  and  $g$ . However, we can rephrase this as the area under  $g$  subtracted from the area under  $f$ , which is given by  $\int_a^b f(x) dx - \int_a^b g(x) dx$ .



Property 4. says that if  $f(x)$  is a positive valued function, then the area underneath the graph of  $f(x)$  between  $a$  and  $b$  plus the area underneath the graph of  $f(x)$  between  $b$  and  $c$  equals the area underneath the graph of  $f(x)$  between  $a$  and  $c$ .

Property 5. follows from Properties 4. and 1. by letting  $c = a$ .

**Example 7:** Using the graph of  $f(x)$  from Example 3, find the integral  $\int_2^5 5f(x) dx$ .

$$\int_2^5 5f(x) dx = 5 \int_2^5 f(x) dx = 5(6) = \boxed{30}$$

property 2 above  
allows us to  
factor out the  
coefficient

answer to  
Example 3(a)  
is 6

**Example 8:** Let

$$\int_1^4 f(x) dx = 3, \quad \int_1^9 f(x) dx = -4, \quad \int_1^4 g(x) dx = 2, \quad \int_1^9 g(x) dx = 8, \quad \int_6^9 g(x) dx = 3.$$

Use these values to evaluate the given definite integrals.

$$\begin{aligned} \text{(a)} \quad \int_1^4 (f(x) - g(x)) dx &= \int_1^4 f(x) dx - \int_1^4 g(x) dx \\ &= 3 - 2 = \boxed{1} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_9^1 (f(x) + g(x)) dx &= - \int_1^9 (f(x) + g(x)) dx \\ &= - \int_1^9 f(x) dx - \int_1^9 g(x) dx = -(-4) - 8 = 4 - 8 = \boxed{-4} \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \int_4^9 (7f(x) + 10g(x)) dx &= 7 \int_4^9 f(x) dx + 10 \int_4^9 g(x) dx \\ &= 7(-7) + 10(6) = -49 + 60 = \boxed{11} \end{aligned}$$

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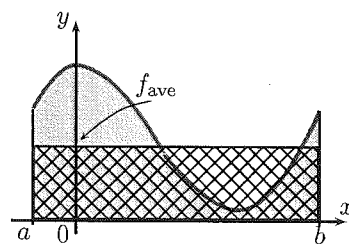
$$\begin{aligned} \text{(d)} \quad \int_4^6 (g(x) - 5) dx &= \int_4^6 g(x) dx - \int_4^6 5 dx = 3 - 10 = \boxed{-7} \end{aligned}$$

see next page! →

► **Average Values:** The average of finitely many numbers  $y_1, y_2, \dots, y_n$  is  $y_{\text{ave}} = \frac{y_1 + y_2 + \dots + y_n}{n}$ . What if we are dealing with infinitely many values? More generally, how can we compute the average of a function  $f$  defined on an interval?

**Average of a function:** The average of a function  $f$  on an interval  $[a, b]$  equals the integral of  $f$  over the interval divided by the length of the interval:

$$f_{\text{ave}} = \frac{\int_a^b f(x) dx}{b - a}.$$



**Geometric meaning:** If  $f$  is a positive valued function,  $f_{\text{ave}}$  is that number such that the rectangle with base  $[a, b]$  and height  $f_{\text{ave}}$  has the same area as the region underneath the graph of  $f$  from  $a$  to  $b$ .

⑧ (c) To find  $\int_4^9 f(x) dx$ :

We know  $\int_1^4 f(x) dx + \int_4^9 f(x) dx = \int_1^9 f(x) dx$

$$\Rightarrow 3 + \int_4^9 f(x) dx = -4$$

$$\Rightarrow \int_4^9 f(x) dx = -4 - 3 = \textcircled{-7}$$

To find  $\int_4^9 g(x) dx$ :

We know  $\int_1^4 g(x) dx + \int_4^9 g(x) dx = \int_1^9 g(x) dx$

$$\Rightarrow 2 + \int_4^9 g(x) dx = 8$$

$$\Rightarrow \int_4^9 g(x) dx = 8 - 2 = \textcircled{6}$$

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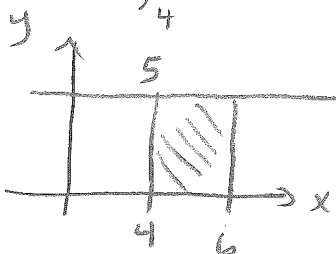
(d) To find  $\int_4^6 g(x) dx$ : We know

$$\int_4^6 g(x) dx + \int_6^9 g(x) dx = \int_4^9 g(x) dx$$

$$\Rightarrow \int_4^6 g(x) dx + 3 = 6 \Rightarrow \int_4^6 g(x) dx = 6 - 3 = \textcircled{3}$$

found in (c)  
above

To find  $\int_4^6 5 dx$ : graph the function  $y = 5$



area of rectangle

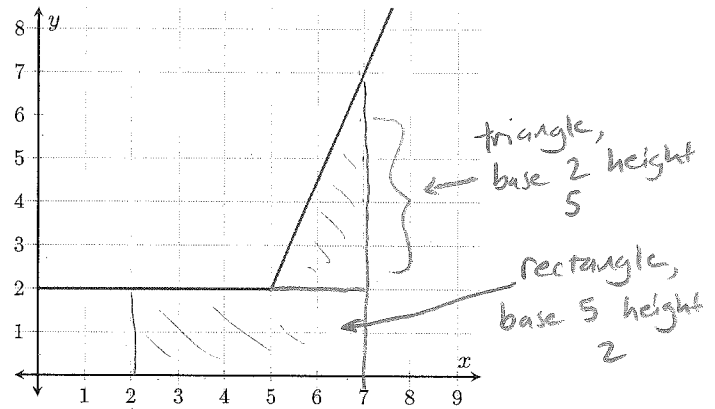
$$= \text{base} \cdot \text{height}$$

$$= 2(5) = \textcircled{10}$$

**Example 9:** Suppose  $f(x) = \begin{cases} 2 & \text{if } x \leq 5 \\ \frac{1}{2}(5x - 21) & \text{if } x > 5. \end{cases}$

Find the average value of  $f(x)$  over the interval  $[2, 7]$ .

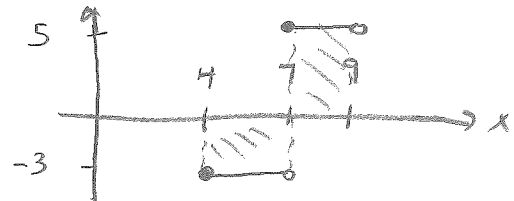
$$\begin{aligned} \int_2^7 f(x) dx &= \text{area of rectangle} + \text{area of triangle} \\ &= 5(2) + \frac{1}{2}(2)(5) \\ &= 10 + 5 = 15. \end{aligned}$$



Average value over  $[2, 7]$

$$= \frac{1}{7-2} \int_2^7 f(x) dx = \frac{1}{5} (15) = \textcircled{3}$$

**Example 10:** Suppose  $f(x) = \begin{cases} -3 & \text{if } 4 \leq x < 7 \\ 5 & \text{if } 7 \leq x \leq 9. \end{cases}$   
 should be  $\leq$  →



(a) Find the average value of  $f(x)$  on the interval  $[4, 9]$ .

$$\int_4^9 f(x) dx = -3(3) + 2(5) = -9 + 10 = 1$$

Average value over  $[4, 9]$

$$= \frac{1}{9-4} \int_4^9 f(x) dx = \frac{1}{5} (1) = \textcircled{\frac{1}{5}}$$

(b) Find the average rate of change of  $f(x)$  on the interval  $[4, 9]$

$$\text{AROC} = \frac{f(9) - f(4)}{9-4} = \frac{5 - (-3)}{5} = \textcircled{\frac{8}{5}}$$