MA123, Chapter 9: Estimating Definite Integrals (pp. 155-176, Gootman)

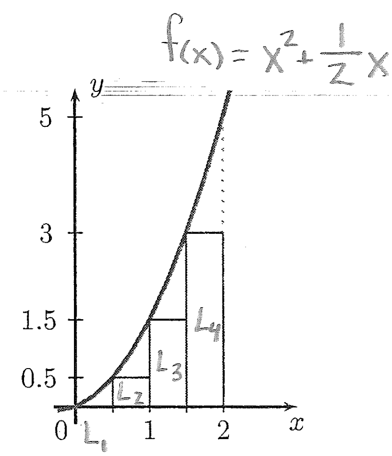
`Chapter Goals:`

- Estimate the value of a definite integral using left endpoints, right endpoints, midpoints, or trapezoids.
- Understand the formal definition of the definite integral. (Optional)

`Assignments:` `Assignment 20` `Assignment 21`

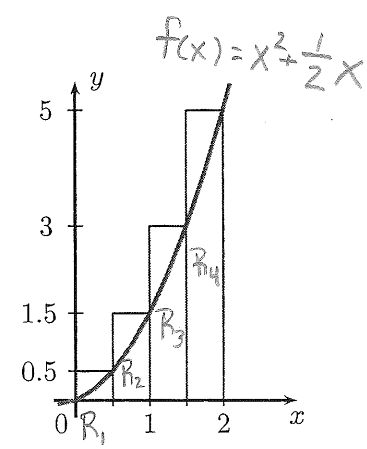
General philosophy: In this chapter, we will learn to estimate the definite integral for functions where we cannot use geometry to compute the areas. *The key idea is to notice that the value of the function does not vary very much over a small interval, and so it is approximately constant over a small interval.* We will use the areas of particular rectangles or trapezoids to estimate the integrals. One particular type of estimation using rectangles is called a Riemann sum.

Example 1: Estimate the area under the graph of $y = x^2 + \frac{1}{2}x$ for x between 0 and 2 in two different ways:



(a) Subdivide the interval $[0, 2]$ into four equal subintervals and use the left endpoint of each subinterval as "sample point."

$$\begin{aligned}\text{Area} &\approx L_1 + L_2 + L_3 + L_4\\ &= f(0)\cdot\tfrac{1}{2} + f(\tfrac{1}{2})\cdot\tfrac{1}{2} + f(1)\cdot\tfrac{1}{2} + f(\tfrac{3}{2})\cdot\tfrac{1}{2}\\ &= 0\cdot\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2} + \tfrac{3}{2}\cdot\tfrac{1}{2} + 3\cdot\tfrac{1}{2}\\ &= 0 + \tfrac{1}{4} + \tfrac{3}{4} + \tfrac{3}{2} = \boxed{\tfrac{5}{2}}\end{aligned}$$



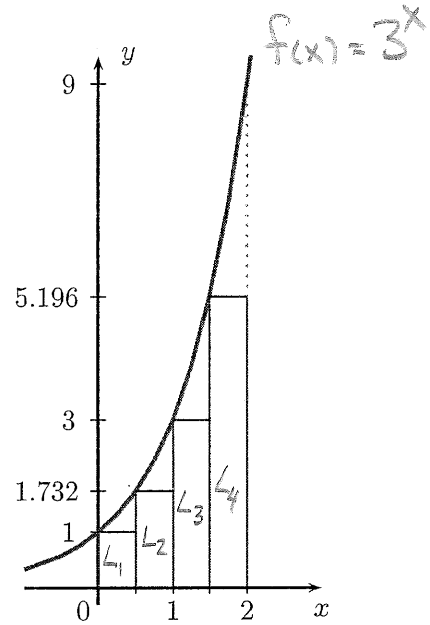
(b) Subdivide the interval $[0, 2]$ into four equal subintervals and use the right endpoint of each subinterval as "sample point."

$$\begin{aligned}\text{Area} &\approx R_1 + R_2 + R_3 + R_4\\ &= f(\tfrac{1}{2})\cdot\tfrac{1}{2} + f(1)\cdot\tfrac{1}{2} + f(\tfrac{3}{2})\cdot\tfrac{1}{2} + f(2)\cdot\tfrac{1}{2}\\ &= \tfrac{1}{2}\cdot\tfrac{1}{2} + \tfrac{3}{2}\cdot\tfrac{1}{2} + 3\cdot\tfrac{1}{2} + 5\cdot\tfrac{1}{2}\\ &= \tfrac{1}{4} + \tfrac{3}{4} + \tfrac{3}{2} + \tfrac{5}{2} = \boxed{5}\end{aligned}$$

Dividing $[0, 2]$ into four equal pieces is an example of a **partition** of the interval $[0, 2]$. Most of our partitions will use equal-width subdivisions, though that is not required.

(c) Find the difference between the two estimates (right endpoint estimate minus left endpoint estimate).

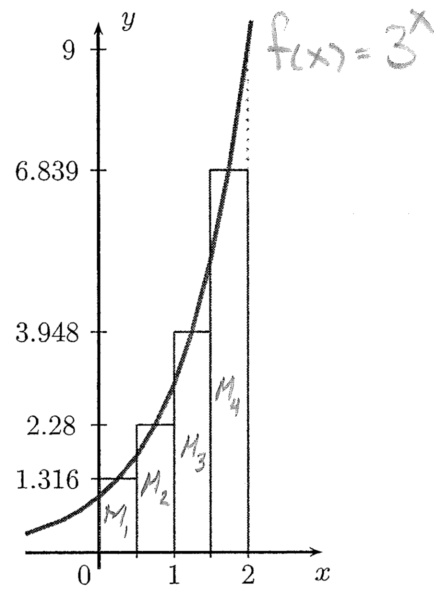
$$5 - \frac{5}{2} = \boxed{\frac{5}{2}}$$



Example 2: Estimate the area under the graph of $y = 3^x$ for x between 0 and 2.

(a) Use a partition that consists of four equal subintervals of $[0, 2]$, and use the left endpoint of each subinterval as a sample point.

$$\text{Area} \approx L_1 + L_2 + L_3 + L_4$$
$$= f(0)\cdot\frac{1}{2} + f(\tfrac{1}{2})\cdot\frac{1}{2} + f(1)\cdot\frac{1}{2} + f(\tfrac{3}{2})\cdot\frac{1}{2}$$
$$= 3^0\cdot\frac{1}{2} + 3^{1/2}\cdot\frac{1}{2} + 3^1\cdot\frac{1}{2} + 3^{3/2}\cdot\frac{1}{2}$$
$$= 1\cdot\frac{1}{2} + \sqrt{3}\cdot\frac{1}{2} + 3\cdot\frac{1}{2} + 3\sqrt{3}\cdot\frac{1}{2}$$
$$= \frac{1}{2} + \frac{\sqrt{3}}{2} + \frac{3}{2} + \frac{3\sqrt{3}}{2} = \boxed{2 + 2\sqrt{3}}$$



(b) Use a partition that consists of four equal subintervals of $[0, 2]$, and use the midpoint of each subinterval as a sample point.

$$\text{Area} \approx M_1 + M_2 + M_3 + M_4$$
$$= f(\tfrac{1}{4})\cdot\frac{1}{2} + f(\tfrac{3}{4})\cdot\frac{1}{2} + f(\tfrac{5}{4})\cdot\frac{1}{2} + f(\tfrac{7}{4})\cdot\frac{1}{2}$$
$$= 3^{1/4}\cdot\frac{1}{2} + 3^{3/4}\cdot\frac{1}{2} + 3^{5/4}\cdot\frac{1}{2} + 3^{7/4}\cdot\frac{1}{2}$$
$$= \sqrt[4]{3}\cdot\frac{1}{2} + \sqrt[4]{27}\cdot\frac{1}{2} + 3\sqrt[4]{3}\cdot\frac{1}{2} + 3\sqrt[4]{27}\cdot\frac{1}{2}$$
$$= \frac{\sqrt[4]{3}}{2} + \frac{\sqrt[4]{27}}{2} + \frac{3\sqrt[4]{3}}{2} + \frac{3\sqrt[4]{27}}{2}$$
$$= \boxed{2\sqrt[4]{3} + 2\sqrt[4]{27}}$$

Note:

In the previous two examples we systematically chose the value of the function at a particular point of each subinterval. However, since the guiding idea is that we are assuming that the values of the function over a small subinterval do not change by very much, then we could take the value of the function at any point of the subinterval as a good sample or representative value of the function. We could also have chosen small subintervals of different lengths. However, we are trying to establish a systematic procedure that works well in general.

Getting better estimates:

We can only expect the previous answers to be approximations of the correct answers. This is because the values of the function do change on each subinterval, even though they do not change by much.

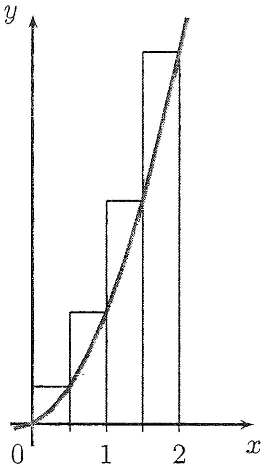
If, however, we replace the subintervals we used by "smaller" subintervals we can reasonably expect the values of the function to vary by much less on each thinner subinterval. Thus, we can reasonably expect that the area of each thinner vertical strip under the graph of the function to be more accurately approximated by the area of these thinner rectangles. Then if we add up the areas of all these thinner rectangles, we should get a much more accurate estimate for the area of the original region.

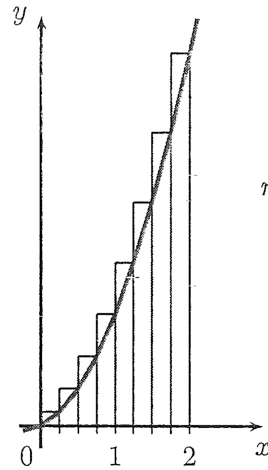
Here is Example 1(*b*), revisited:

$y = x^2 + \frac{1}{2}x$ on $[0, 2]$

$n = 4$ equal subintervals

Area ≈ 5





$y = x^2 + \frac{1}{2}x$ on $[0, 2]$

$n = 8$ equal subintervals

Area ≈ 4.3125

We will see later that the exact value of the area under consideration in Example 1 is $\frac{11}{3} \approx 3.66$.

Example 3: The rate (in liters per minute) at which water drains from a tank is recorded at two-minute intervals. Use the average of the left- and right-endpoint approximations to estimate the total amount of water drained during the first 6 min.

t min	0	2	4	6
l/min	48	46	44	42

Left Endpoint approx

$= 48 \cdot 2 + 46 \cdot 2 + 44 \cdot 2$

$= 96 + 92 + 88$

$= 276$ liters

Right Endpoint approx

$= 46 \cdot 2 + 44 \cdot 2 + 42 \cdot 2$

$= 92 + 88 + 84$

$= 264$ liters

Average of left and right endpoint approx

$= \frac{276 + 264}{2}$

$= \boxed{270 \text{ liters}}$

Example 4: We could estimate the area under the graph of $y = \frac{1}{x}$ for x between 1 and 31 by dividing the interval $[1, 31]$ into 30 equal subintervals and using the left endpoint of each subinterval as sample point. Next, we could estimate the area using the right endpoint as sample point.
Find the difference between the two estimates (left endpoint estimate minus right endpoint estimate).

$$\Delta x = \frac{31-1}{30} = \frac{30}{30} = 1$$

$$\text{Left endpoint approx} = f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 + \cdots + f(30)\cdot 1$$
$$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{30}$$

$$\text{Right endpoint approx} = f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1 + \cdots + f(30)\cdot 1 + f(31)\cdot 1$$
$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{30} + \frac{1}{31}$$

$$\text{Difference} = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{30}\right) - \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{30} + \frac{1}{31}\right) = 1 - \frac{1}{31} = \boxed{\frac{30}{31}}$$

Example 5: Suppose you estimate the area under the graph of $f(x) = x^3$ from $x = 4$ to $x = 24$ by adding the areas of rectangles as follows: partition the interval into 20 equal subintervals and use the right endpoint of each interval to determine the height of the rectangle. What is the area of the 15^{th} rectangle?

$$\Delta x = \frac{24-4}{20} = \frac{20}{20} = 1$$

$$\text{Right endpoint approx} = \underbrace{f(5)\cdot 1}_{1^{st}\text{ rectangle}} + \overbrace{f(6)\cdot 1}^{2^{nd}\text{ rectangle}} + f(7)\cdot 1 + \cdots + f(24)\cdot 1$$

$$\Rightarrow 15^{th} \text{ rectangle has area} = f(19)\cdot 1 = 19^3\cdot 1 = \boxed{6859}$$

Example 6: Suppose you estimate the area under the graph of $f(x) = \frac{1}{x}$ from $x = 12$ to $x = 112$ by adding the areas of rectangles as follows: partition the interval into 50 equal subintervals and use the left endpoint of each interval to determine the height of the rectangle. What is the area of the 24^{th} rectangle?

$$\Delta x = \frac{112-12}{50} = \frac{100}{50} = 2$$

$$\text{Left endpoint approx} = \underbrace{f(12)\cdot 2}_{1^{st}\text{rectangle}} + \overbrace{f(14)\cdot 2}^{2^{nd}\text{ rectangle}} + f(16)\cdot 2 + \cdots + f(110)\cdot 2$$

$$\Rightarrow 24^{th} \text{ rectangle has area} = f(58)\cdot 2 = \frac{1}{58}\cdot 2 = \boxed{\frac{1}{29}}$$

Example 7: An object travels in a straight line, and we would like to estimate how far the object traveled during the time interval $0 \le t \le 5$, but we only have the following information about the velocity of the object:

time (sec)	0	1	2	3	4	5
velocity (m/sec)	-3	-1	-4	1	3	6

(a) Using left endpoints as sample points, estimate the object's total displacement over the time interval.

$$\text{Displacement} = (-3)\cdot 1 + (-1)\cdot 1 + (-4)\cdot 1 + 1\cdot 1 + 3\cdot 1$$
$$= -3 - 1 - 4 + 1 + 3$$
$$= -4 \text{ meters}$$

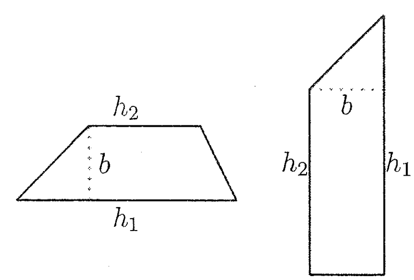
(b) Using left endpoints as sample points, estimate the object's total distance traveled over the time interval.

$$\text{Distance Traveled} = |-3|\cdot 1 + |-1|\cdot 1 + |-4|\cdot 1 + |1|\cdot 1 + |3|\cdot 1$$
$$= 3\cdot 1 + 1\cdot 1 + 4\cdot 1 + 1\cdot 1 + 3\cdot 1$$
$$= 12 \text{ meters}$$

Trapezoids versus rectangles:

We could use trapezoids instead of rectangles to obtain better estimates, even though the calculations get a little bit more complicated. This will occur in some of the later examples. We recall that the area of a trapezoid is

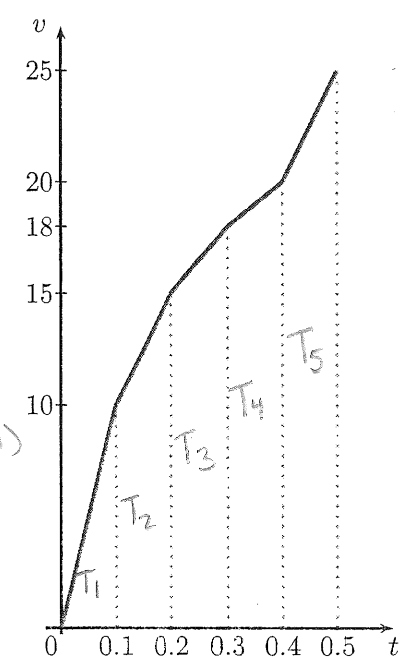
$$\text{Area of a trapezoid} = \frac{(h_1 + h_2)\cdot b}{2}.$$



Example 8: A train travels in a straight westward direction along a track. The velocity of the train varies, but it is measured at regular time intervals of 1/10 hour. The measurements for the first half hour are

time	0	0.1	0.2	0.3	0.4	0.5
velocity	0	10	15	18	20	25

where the velocity in the table is given in miles per hour. Compute the total distance traveled by the train during the first half hour by assuming the velocity is a linear function of t on the subintervals.

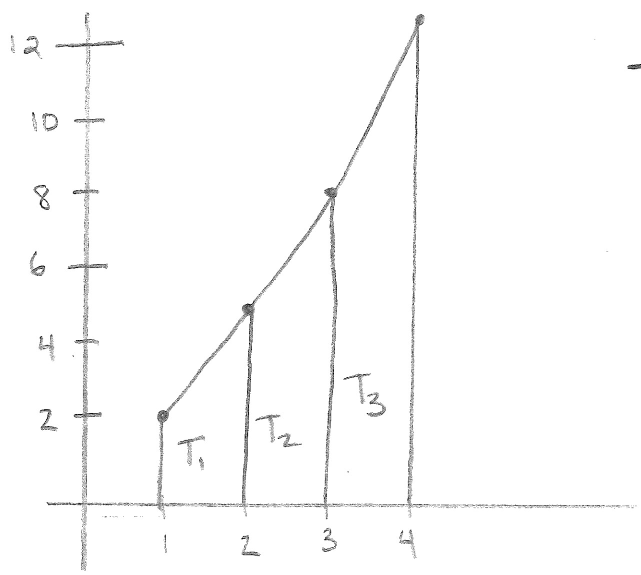


$$\text{Distance} = T_1 + T_2 + T_3 + T_4 + T_5$$
$$= \frac{(0+10)(0.1)}{2} + \frac{(10+15)(0.1)}{2} + \frac{(15+18)(0.1)}{2} + \frac{(18+20)(0.1)}{2} + \frac{(20+25)(0.1)}{2}$$
$$= 0.5 + 1.25 + 1.65 + 1.9 + 2.25$$
$$= \boxed{7.55 \text{ miles}}$$

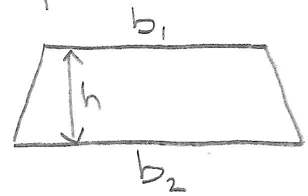
Example 9: Suppose you are given the following data points for a function $f(x)$:

x	1	2	3	4
$f(x)$	2	5	8	12

If f is a linear function on each interval between the given points, find $\int_1^4 f(x)\,dx$.



– T_1, T_2, and T_3 are Trapezoids.



The area of a trapezoid is given by $A = \frac{h}{2}(b_1 + b_2)$

$$\begin{aligned}
\int_1^4 f(x)\,dx &= T_1 + T_2 + T_3 \\
&= \frac{1}{2}(2+5) + \frac{1}{2}(5+8) + \frac{1}{2}(8+12) \\
&= \frac{7}{2} + \frac{13}{2} + \frac{20}{2} \\
&= \boxed{20}
\end{aligned}$$