MA123, Supplement: Exponential and logarithmic functions (pp. 315-319, Gootman)

`Chapter Goals:`
- Review properties of exponential and logarithmic functions.
- Learn how to differentiate exponential and logarithmic functions.
- Learn about exponential growth and decay phenomena.

`Assignments:` `Assignment 10` `Assignment 11`

Quick review

Exponential notation: If a is any real number and n is a positive integer, then the **n-th power** of a is

$$a^n = \underbrace{a \cdot a \cdot \ \dots \ \cdot a}_{n \text{ times}}$$

The number a is called the **base** whereas n is called the **exponent**.

The first and second laws of exponents below allow us to define a^n for any integer n.

Now, we want to define, for instance, $a^{1/3}$ in a way that is consistent with the laws of exponents. We would like:

$$\left(a^{1/3}\right)^3 = a^{(1/3)3} = a^1 = a; \qquad \text{thus} \qquad a^{1/3} = \sqrt[3]{a}$$

So, by the definition of nth root, we have:

$$\boxed{a^{1/n} = \sqrt[n]{a}}$$

Definition of rational exponents: For any rational exponent m/n in lowest terms, where m and n are integers and $\boxed{n > 0}$, we define

This should be n>1?

$$a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m \qquad \text{or equivalently}$$

$$a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}$$

If n is even we require that $a \geq 0$.

In the table below, a and b are real numbers ($\neq 0$ if needed) and the exponents x and y are rational numbers.

Laws of exponents:

(1.) $a^0 = 1$

(2.) $a^{-x} = \dfrac{1}{a^x}$

(3.) $a^x a^y = a^{x+y}$

(4.) $\dfrac{a^x}{a^y} = a^{x-y}$

(5.) $(a^x)^y = a^{xy}$

(6.) $(ab)^x = a^x b^x$

(7.) $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

Now, let $a > 0$ be a positive number with $a \neq 1$. Thus far a^x is defined for x a rational number. So, what does, for instance, $5^{\sqrt{2}}$ mean? When x is irrational, we successively approximate x by rational numbers. For instance, as

$$\sqrt{2} \approx 1.41421\dots$$

we successively approximate $5^{\sqrt{2}}$ with

$$5^{1.4}, \quad 5^{1.41}, \quad 5^{1.414}, \quad 5^{1.4142}, \quad 5^{1.41421}, \dots$$

In practice, we simply use our calculator and find out

$$5^{\sqrt{2}} \approx 9.73851\dots$$

▶ **Exponential functions:**

Let $a > 0$ be a positive number with $a \neq 1$. The **exponential function with base** a is defined by

$$f(x) = a^x$$

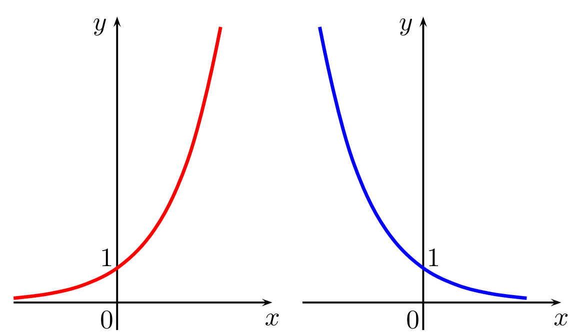
for all real numbers x.

Graphs of exponential functions:

The **exponential function**

$$f(x) = a^x \qquad (a > 0,\ a \neq 1)$$

has domain $\mathbb{R}$ and range $(0, \infty)$. The graph of $f(x)$ has one of these shapes:



$f(x) = a^x$ for $a > 1$ $\qquad$ $f(x) = a^x$ for $0 < a < 1$

The most important base is the number denoted by the letter e. The number e is defined as

$$\boxed{e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}$$

Correct to five decimal places (note that e is an irrational number), $e \approx 2.71828$.

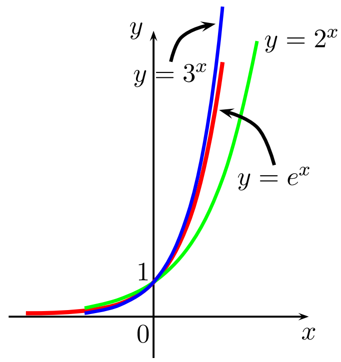
The natural exponential function:

The **natural exponential function** is the exponential function

$$f(x) = e^x$$

with base e. It is often referred to as _the_ exponential function.

Since $2 < e < 3$, the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$.



n	$\left(1+\frac{1}{n}\right)^n$
1	2.00000
5	2.48832
10	2.59374
100	2.70481
1,000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

► **Logarithmic functions:** Every exponential function $f(x) = a^x$, with $a > 0$ and $a \neq 1$, is a one-to-one function by the *horizontal line test*. Thus, it has an inverse function. The inverse function $f^{-1}(x)$ is called the *logarithmic function with base a* and is denoted by $\log_a x$.

Definition: Let a be a positive number with $a \neq 1$. The **logarithmic function** with base a, denoted by $\log_a$, is defined by

$$y = \log_a(x) \iff a^y = x.$$

In other words, $\log_a(x)$ is the exponent to which the base a must be raised to give x.

Properties of logarithms:

(1.) $\log_a(1) = 0$ (3.) $\log_a(a^x) = x$

(2.) $\log_a(a) = 1$ (4.) $a^{\log_a(x)} = x$

Since logarithms are 'exponents', the laws of exponents give rise to the laws of logarithms:

Let a be a positive number, with $a \neq 1$. Let A, B and C be any real numbers with $A > 0$ and $B > 0$.

Laws of logarithms:

(1.) $\log_a(AB) = \log_a(A) + \log_a(B)$;

(2.) $\log_a\left(\frac{A}{B}\right) = \log_a(A) - \log_a(B)$;

(3.) $\log_a(A^C) = C\,\log_a(A)$.

Change of base:

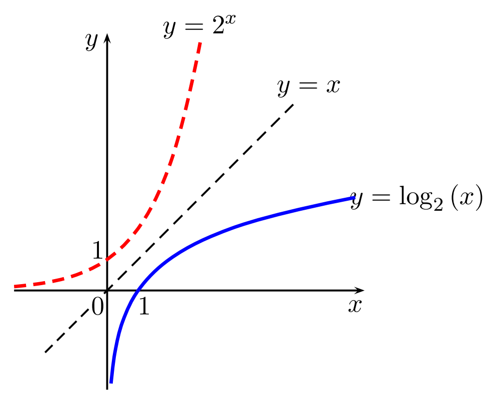
For some purposes, we find it useful to change from logarithms in one base to logarithms in another base. One can prove that:

$$\boxed{\log_b x = \frac{\log_a(x)}{\log_a(b)}}$$

Remark: If a one-to-one function f has domain A and range B, then its inverse function f^{-1} has domain B and range A. THUS, the function $y = \log_a(x)$ is defined for $x > 0$ and has range equal to $\mathbb{R}$. More precisely:

Graphs of logarithmic functions:

The graph of $f^{-1}(x) = \log_a(x)$ is obtained by reflecting the graph of $f(x) = a^x$ in the line $y = x$.
(The picture below shows a typical case with $a > 1$.)



The point $(1, 0)$ is on the graph of $y = \log_a(x)$ (as $\log_a(1) = 0$) and the y-axis is a vertical asymptote.

Common logarithms:

The logarithm with base 10 is called the **common logarithm** and is denoted by omitting the base:

$$\boxed{\log(x) := \log_{10}(x).}$$

Natural logarithms: Of all possible bases a for logarithms, it turns out that the most convenient choice for the purposes of Calculus is the number e.

Definition: The logarithm with base e is called the **natural logarithm** and is denoted by ln:

$$\boxed{\ln(x) := \log_e(x).}$$

We recall again that, by the definition of inverse functions, we have

$$y = \ln(x) \qquad \Longleftrightarrow \qquad e^y = x.$$

Properties of natural logarithms:

(1.) $\ln(1) = 0$

(2.) $\ln(e) = 1$

(3.) $\ln(e^x) = x$

(4.) $e^{\ln(x)} = x$

Derivatives

Fact: By filling the table below we can convince ourselves that $\boxed{\lim_{h\to 0} \frac{e^h - 1}{h} = 1.}$

h	-0.1	-0.01	-0.001	-0.0001	-0.00001	0.00001	0.0001	0.001	0.01	0.1
$\frac{e^h-1}{h}$	0.9516	0.9950	0.9995	0.9999	0.9999	1.0000	1.0000	1.0005	1.0050	1.0517

Now, let $f(x) = e^x$. Using the definition of the derivative and the rules for exponential functions, we have that

$$\frac{d}{dx}(e^x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x e^h - e^x}{h} = e^x \left(\lim_{h\to 0} \frac{e^h - 1}{h} \right) = e^x$$

Theorem: $$\boxed{\frac{d}{dx}(e^x) = e^x \qquad \text{or} \qquad (e^x)' = e^x.}$$

Moreover, it follows by applying the chain rule that

$$\boxed{\frac{d}{dx}(e^{g(x)}) = e^{g(x)} \frac{d}{dx}(g(x)) \qquad \text{or} \qquad (e^{g(x)})' = e^{g(x)}\, g'(x).}$$

We can use the derivative of e^x and the relationship between the exponential and the natural logarithmic functions to find the derivative of the function $\ln(x)$. Namely, take the derivative with respect to x of both sides of $e^{\ln(x)} = x$. We obtain

x

$$\frac{d}{dx}(e^{\ln x}) = \frac{d}{dx}(x) \qquad \text{or} \qquad e^{\ln x}\frac{d}{dx}(\ln x) = 1 \qquad \text{or} \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Theorem: $$\boxed{\frac{d}{dx}(\ln(x)) = \frac{1}{x} \qquad \text{or} \qquad (\ln(x))' = \frac{1}{x}.}$$

Moreover, it follows by applying the chain rule that

$$\boxed{\frac{d}{dx}(\ln(g(x))) = \frac{1}{g(x)}\frac{d}{dx}(g(x)) \qquad \text{or} \qquad (\ln(g(x)))' = \frac{g'(x)}{g(x)}.}$$

What about more general derivatives? Observe that we have the identities

$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x\ln(a)}) = e^{x\ln(a)} \cdot \frac{d}{dx}(x\ln(a)) = e^{x\ln(a)} \cdot \ln(a) = \boxed{a^x \ln(a)}$

$$a^x = e^{\ln(a^x)} = e^{x\ln(a)} \qquad \log_a(x) = \frac{\ln(x)}{\ln(a)}.$$

Change of base formula

$\frac{d}{dx}(\log_a(x)) = \frac{d}{dx}\left(\frac{\ln(x)}{\ln(a)}\right) = \frac{d}{dx}\left(\frac{1}{\ln(a)} \cdot \ln(x)\right) = \frac{1}{\ln(a)} \cdot \frac{d}{dx}(\ln(x)) = \frac{1}{\ln(a)} \cdot \frac{1}{x} = \boxed{\frac{1}{x\ln(a)}}$

Thus using the previous results we obtain the following formulas for the derivatives of general exponential and logarithmic functions

$$\boxed{\frac{d}{dx}(a^x) = a^x \ln(a) \qquad \text{and} \qquad \frac{d}{dx}(\log_a(x)) = \frac{1}{x\ln(a)}.}$$

Note: Let us consider the function $f(x) = 3^x$. In Example 16 of Chapter 4, we saw that an approximation for $f'(1)$ was given by the value 3.2958. Using the above formula we have that $f'(x) = 3^x \ln(3)$, so that the exact value for $f'(1)$ is $3\ln(3) = \ln(27)$.

Example 1: Find the derivative with respect to x of $f(x) = e^{4x}$. Evaluate $f'(x)$ at $x = 1/4$. Compute $f''(x)$, $f'''(x)$ and $f^{(10)}(x)$. Can you guess what the nth derivative $f^{(n)}(x)$ of $f(x)$ looks like?

Recall: If $f(x) = e^{g(x)}$ then $f'(x) = e^{g(x)} \cdot \frac{d}{dx}(g(x))$

$f(x) = e^{4x}$

$f'(x) = e^{4x} \cdot \frac{d}{dx}(4x) = 4e^{4x}$

So $f'(\frac{1}{4}) = 4e^{4 \cdot \frac{1}{4}} = 4e$

$f''(x) = \frac{d}{dx}(4e^{4x}) = 4\frac{d}{dx}(e^{4x}) = 4 \cdot 4e^{4x} = 4^2 e^{4x}$

$f'''(x) = \frac{d}{dx}(4^2 e^{4x}) = 4^2 \frac{d}{dx}(e^{4x}) = 4^2 \cdot 4e^{4x} = 4^3 e^{4x}$

so $f^{(10)}(x) = 4^{10} e^{4x}$

and $f^{(n)}(x) = 4^n e^{4x}$

Example 2: Find the derivative with respect to x of $g(x) = x^2 e^x$. Evaluate $g'(x)$ at $x = 1$.

$g'(x) = \frac{d}{dx}(x^2 \cdot e^x) = \frac{d}{dx}(x^2) \cdot e^x + x^2 \cdot \frac{d}{dx}(e^x) = 2xe^x + x^2 e^x$

So $g'(1) = 2(1)e^1 + 1^2 e^1 = 2e + e = 3e$

Example 3: Suppose $f(t) = e^{\sqrt{3t-4}}$. Find $\frac{df}{dt}$.

Recall: If $f(x) = e^{g(x)}$ then $\frac{df}{dx} = e^{g(x)} \cdot \frac{d}{dx}(g(x))$

So $f(t) = e^{\sqrt{3t-4}}$ implies that $\frac{df}{dt} = e^{\sqrt{3t-4}} \cdot \frac{d}{dt}(\sqrt{3t-4}) = e^{\sqrt{3t-4}} \cdot \frac{d}{dt}\left((3t-4)^{\frac{1}{2}}\right)$

$= e^{\sqrt{3t-4}} \cdot \frac{1}{2}(3t-4)^{\frac{1}{2}-1} \cdot \frac{d}{dt}(3t-4) = e^{\sqrt{3t-4}} \cdot \frac{1}{2}(3t-4)^{-\frac{1}{2}} \cdot 3 = \boxed{\frac{3e^{\sqrt{3t-4}}}{2\sqrt{3t-4}}}$

Chain Rule

Example 4: Find the derivative with respect to x of $y = \ln(e^x)$.

Note $y = \ln(e^x) = x$

So $\frac{dy}{dx} = 1$

Example 5: Find the derivative with respect to x of $f(x) = x\ln(x)$.

$$f'(x) = \frac{d}{dx}(x\ln(x)) = \frac{d}{dx}(x)\cdot\ln(x) + x\,\frac{d}{dx}(\ln(x))$$
$$= 1\cdot\ln(x) + x\cdot\frac{1}{x}$$
$$= \ln(x) + 1$$

Example 6: Find the derivative with respect to x of $y = \ln(5x+1)$.

Recall $\frac{d}{dx}(\ln(g(x))) = \dfrac{\frac{dg}{dx}}{g(x)}$

$$\frac{dy}{dx} = \frac{d}{dx}(\ln(5x+1)) = \frac{\frac{d}{dx}(5x+1)}{5x+1} = \boxed{\frac{5}{5x+1}}$$

Example 7: Find $\frac{d}{dx}\Big(\ln(3x^4 - 7x^2 + 5)\Big)$.

Recall $\frac{d}{dx}(\ln(g(x))) = \dfrac{\frac{dg}{dx}}{g(x)}$

So $$\frac{d}{dx}(\ln(3x^4-7x^2+5)) = \frac{\frac{d}{dx}(3x^4-7x^2+5)}{3x^4-7x^2+5} = \frac{3\cdot 4x^3 - 7\cdot 2x}{3x^4-7x^2+5} = \boxed{\frac{12x^3-14x}{3x^4-7x^2+5}}$$

Example 8: Find the derivative with respect to x of $f(x) = \ln(\ln(\ln(x)))$.

$$\frac{d}{dx}\Big[\ln(\ln(\ln(x)))\Big] = \frac{\frac{d}{dx}\big[\ln(\ln(x))\big]}{\ln(\ln(x))} = \frac{\dfrac{\frac{d}{dx}[\ln(x)]}{\ln(x)}}{\ln(\ln(x))}$$
$$= \frac{\frac{d}{dx}(\ln(x))}{\ln(x)}\cdot\frac{1}{\ln(\ln(x))} = \frac{\frac{1}{x}}{\ln(x)\ln(\ln(x))} = \frac{1}{x}\cdot\frac{1}{\ln(x)\ln(\ln(x))} = \boxed{\frac{1}{x\ln(x)\ln(\ln(x))}}$$

Example 9: Find the derivative with respect to x of $h(x) = e^{x^2+3\ln(x)}$.

Recall $\frac{d}{dx}(e^{g(x)}) = e^{g(x)}\cdot\frac{d}{dx}(g(x))$

So $$\frac{dh}{dx} = \frac{d}{dx}(e^{x^2+3\ln(x)}) = e^{x^2+3\ln(x)}\cdot\frac{d}{dx}(x^2+3\ln(x))$$
$$= e^{x^2+3\ln(x)}\cdot\left[2x + 3\cdot\frac{1}{x}\right] = \boxed{\left(2x+\frac{3}{x}\right)e^{x^2+3\ln(x)}}$$

Exponential growth and decay

Let $Q(t)$ denote the amount of a quantity as a function of time. We say that $Q(t)$ grows exponentially as a function of time if

$$Q(t) = Q_0\, e^{rt},$$

where Q_0 and r are positive constants that depend on the specific problem and t denotes time. When $t = 0$, we see that

$$Q(0) = Q_0\, e^{r \cdot 0} = Q_0 \cdot 1 = Q_0.$$

Thus Q_0 denotes the amount of the quantity at $t = 0$. In other words, Q_0 is the initial amount of the quantity at time $t = 0$. Note that $Q(t) > 0$ because $Q_0 > 0$ and $e^{rt} > 0$.

Taking the derivative and using the chain rule, we see that

$$Q'(t) = Q_0 \cdot r \cdot e^{rt} = r\,(Q_0\, e^{rt}) = r\, Q(t).$$

Since $\boxed{Q'(t) = r\, Q(t),}$ it follows that if a quantity grows exponentially, then its rate of growth is proportional to the quantity present, and the proportionality constant is given by r. Since $r > 0$ and $Q(t) > 0$, we have $Q'(t) > 0$, as expected because $Q(t)$ is increasing.

Some quantities decrease exponentially. In this case we have $Q(t) = Q_0\, e^{-rt}$, where Q_0 and r are positive constants. Note that we have $Q(0) = Q_0$ and

$$Q'(t) = Q_0 \cdot (-r) \cdot e^{-rt} = -r\,(Q_0\, e^{-rt}) = -r\, Q(t).$$

Thus $\boxed{Q'(t) = -r\, Q(t).}$ We see that $Q'(t) < 0$ because $-r < 0$ and $Q(t) > 0$. Thus the rate of increase of $Q(t)$ is proportional to the quantity present, and the proportionality constant is given by $-r$.

Suppose that a function $g(x)$ satisfies the property that the slope of the tangent line to the graph of $y = g(x)$ at any point P is proportional to the y-coordinate of P, i.e., $g'(x_P) = r \cdot g(x_P)$. Then it can be shown that there are constants C and r such that $g(x) = Ce^{rx}$. In fact, r is the constant of proportionality because $g'(x) = rCe^{rx} = rg(x)$.

Example 10: The graph of a function $g(x)$ passes through the point $(0, 5)$. Suppose that the slope of the tangent line to the graph of $y = g(x)$ at any point P is 7 times the y-coordinate of P. Find $g(2)$.

$g'(x) = 7g(x)$

so $g(x) = Ce^{7x}$

Note: $5 = g(0) = Ce^{7(0)} = Ce^0 = C$ so $C = 5$ and we have

$g(x) = 5e^{7x}$

$g(2) = 5e^{7\cdot 2} = \boxed{5e^{14}}$

Applications

Many processes that occur in nature, such as calculation of bank interest, population growth, radioactive decay, heat diffusion, and numerous others, can be modeled using exponential functions. Logarithmic functions are used in models for the loudness of sounds, the intensity of earthquakes, and many other phenomena.

Compound interest is calculated by the formula:

$$\boxed{P(t) = P_0\left(1+\frac{r}{n}\right)^{nt}}$$

where

$P(t)$ = principal after t years
P_0 = initial principal
r = interest rate per year
n = number of times interest is compounded per year
t = number of years

Continuously compounded interest is calculated by the formula:

$$\boxed{P(t) = P_0\, e^{rt}}$$

where

$P(t)$ = principal after t years
P_0 = initial principal
r = interest rate per year
t = number of years

Proof: The interest paid increases as the number n of compounding periods increases. If $m = \dfrac{n}{r}$, then:

$$P_0\left(1+\frac{r}{n}\right)^{nt} = P_0\left[\left(1+\frac{r}{n}\right)^{n/r}\right]^{rt} = P_0\left[\left(1+\frac{1}{n/r}\right)^{n/r}\right]^{rt} = P_0\left[\left(1+\frac{1}{m}\right)^{m}\right]^{rt}.$$

As n becomes large, m also becomes large. Since $\lim_{m\to\infty}\left(1+\dfrac{1}{m}\right)^m = e$ we obtain the formula for continuously compounded interest.

Example 11: If $\underbrace{\boxed{\$10,000 \text{ is invested}}}^{P_0}$ at an $\overbrace{\boxed{\text{interest rate of } 6\%}}^{r}$, find the value of the investment at the end of $\underbrace{\boxed{8 \text{ years}}}_{t}$ if the interest is compounded continuously.

$P_0 = 10,000$
$r = 0.06$
$t = 8$

$P(t) = P_0 e^{rt}$
$P(t) = 10,000\, e^{0.06t}$

So $P(8) = 10,000\, e^{0.06\cdot 8}$
$P(8) = 10,000\, e^{0.48}$
$P(8) = \boxed{16,160.74}$

Example 12: How many years will it take an investment $\underbrace{\boxed{\text{to quadruple}}}_{4 \text{ times}}$ in value if the interest is compounded continuously at a rate of 7%?

$P(t) = P_0 e^{rt}$
$r = 0.07$
So $P(t) = P_0 e^{0.07t}$

Initial Principal $= P_0$
4 times Initial Principal $= 4P_0$
So one needs the value of t such that $P(t) = 4P_0$
Consequently, replacing $P(t)$ with $P_0 e^{0.07t}$ yields
$P_0 e^{0.07t} = 4P_0$

$P_0 e^{0.07t} = 4P_0$ Divide by P_0
$\dfrac{\cancel{P_0} e^{0.07t}}{\cancel{P_0}} = \dfrac{4\cancel{P_0}}{\cancel{P_0}}$ Simplify
$e^{0.07t} = 4$ Apply ln
$\ln(e^{0.07t}) = \ln(4)$ Use $\ln(e^x) = x$
$0.07t = \ln(4)$ Divide by 0.07
$\dfrac{\cancel{0.07}t}{\cancel{0.07}} = \dfrac{\ln(4)}{0.07}$ Simplify
$t = \dfrac{\ln(4)}{0.07} \approx \boxed{19.8 \text{ years}}$

Example 13: An amount of P_0 dollars is invested at 5% interest compounded continuously. Find P_0 if at the end of 10 years the value of the investment is $20,000.

$r=0.05$ $t=10$

$P(t)=P_0e^{rt}$

So $P(t)=P_0e^{0.05t}$ $P(10)=20{,}000$

By the above $20{,}000=P(10)=P_0e^{0.05(10)}$

$20{,}000=P_0e^{0.05(10)}$ Divide by $e^{0.05(10)}$

$\frac{20{,}000}{e^{0.05(10)}}=\frac{P_0\cancel{e^{0.05(10)}}}{\cancel{e^{0.05(10)}}}$ Simplify

$P_0=\frac{20{,}000}{e^{0.05(10)}}$

$\boxed{P_0\approx \$12{,}130.61}$

Exponential models of population growth:

The formula for population growth of several species is the same as that for continuously compounded interest. In fact in both cases the rate of growth of a population (or an investment) per time period is proportional to the size of the population (or the amount of an investment).

Exponential growth model If P_0 is the initial size of a population that experiences **exponential growth**, then the population $P(t)$ at time t increases according to the model

$$\boxed{P(t)=P_0e^{rt}}$$

where r is the relative rate of growth of the population (expressed as a proportion of the population).

Remark: Biologists sometimes express the growth rate r in terms of the **doubling-time** t_0, the time required for the population to double in size: $r=\dfrac{\ln(2)}{t_0}$.

Note: If t_0 denotes the doubling-time of a population, we can rewrite the expression for $P(t)$ as follows

$$P(t)=P_0\,e^{rt}=P_0\,e^{(\ln(2)/t_0)\cdot t}=P_0\left(e^{\ln(2)}\right)^{t/t_0}=P_0\,2^{t/t_0}.$$

Example 14: A bacteria culture starts with 2,000 bacteria and the population triples after 5 hours. If we express the number of bacteria after t hours as

$$y(t)=a\,e^{bt}$$

find a and b.

Initial Population, $\boxed{a=2{,}000}$

Note triple population $=3\cdot(2{,}000)=6{,}000$

So $y(5)=6{,}000$

$6{,}000=y(5)=2000\,e^{b(5)}$

$6{,}000=2000\,e^{5b}$ Divide by 2,000

$\frac{6{,}000}{2{,}000}=\frac{\cancel{2{,}000}\,e^{5b}}{\cancel{2{,}000}}$ Simplify

$3=e^{5b}$ Apply ln

$\ln(3)=\ln(e^{5b})$ Use $\ln(e^x)=x$

$\ln(3)=5b$ Divide by 5

$\frac{\ln(3)}{5}=\frac{5b}{5}$ Simplify

$\boxed{b=\frac{\ln(3)}{5}}$

Example 15: A bacteria culture starts with 5,000 bacteria and the population quadruples after 3 hours. Find an expression for the number $P(t)$ of bacteria after t hours.

Note: Initial Population $=5{,}000$

Quadruple population $=4\cdot 5{,}000$
$=20{,}000$

So $P(3)=20{,}000$

$P_0=5{,}000$

So $P(t)=5{,}000e^{rt}$ → $\boxed{P(t)=5{,}000\,e^{\frac{\ln(4)}{3}t}}$

$20{,}000=P(3)=5{,}000\,e^{r\cdot 3}$

$20{,}000=5{,}000\,e^{3r}$ Divide by 5,000

$\frac{20{,}000}{5{,}000}=\frac{5{,}000e^{3r}}{5{,}000}$ Simplify

$4=e^{3r}$ Apply ln

$\ln(4)=\ln(e^{3r})$ Use $\ln(e^x)=x$

$\ln(4)=3r$ Divide by 3

$\frac{\ln(4)}{3}=\frac{\cancel{3}r}{\cancel{3}}$ Simplify

$\boxed{r=\frac{\ln(4)}{3}}$

58

Example 16: If the bacteria in a culture doubles in 3 hours, how many hours will it take before 7 times the original number is present? Recall $r = \frac{\ln(2)}{t_0}$ where t_0 is the doubling time

$P(t) = P_0 e^{\frac{\ln(2)}{3} t}$ So $r = \frac{\ln(2)}{3}$

Initial population $= P_0$

7 times Initial Population $= 7P_0$

So one needs the value of t when

$P(t) = 7P_0$

Consequently, replacing $P(t)$ with $P_0 e^{\frac{\ln(2)}{3} t}$ yields

$7P_0 = P_0 e^{\frac{\ln(2)}{3} t}$ Divide by P_0

$\frac{7\cancel{P_0}}{\cancel{P_0}} = \frac{\cancel{P_0} e^{\frac{\ln(2)}{3} t}}{\cancel{P_0}}$ Simplify

$7 = e^{\frac{\ln(2)}{3} t}$ Apply ln

$\ln(7) = \ln\left(e^{\frac{\ln(2)}{3} t}\right)$ Use $\ln(e^x) = x$

$\ln(7) = \frac{\ln(2)}{3} t$ multiply by $\frac{3}{\ln(2)}$

$\frac{3}{\ln(2)} \cdot \ln(7) = \frac{\cancel{3}}{\cancel{\ln(2)}} \cdot \frac{\cancel{\ln(2)}}{\cancel{3}} t$ Simplify

$\frac{3\ln(7)}{\ln(2)} = t$

$t = \frac{3\ln(7)}{\ln(2)} = \boxed{8.422 \text{ hours}}$

Example 17: If the world population in 2010 was 6 billion and it were to grow exponentially with a growth constant $r = \frac{1}{30}\ln(2)$, find the population (in billions) in the year 2070.

Let initial time $t = 0$ be 2010

then $t = 60$ would be 2070

So $P(60) = 6e^{\frac{\ln(2)}{30} \cdot 60} = \boxed{24 \text{ billion}}$

Radioactive decay:

Radioactive substances decay by spontaneously emitting radiation. In this situation, the rate of decay is proportional to the mass of the substance.

This is analogous to population growth, except that the quantity of radioactive material *decreases*.

Remark: Physicists sometimes express the rate of decay in terms of the **half-life**, the time required for half the mass to decay.

Radioactive decay model:

If Q_0 is the initial quantity of a radioactive substance with half-life t_0, then the quantity $Q(t)$ remaining at time t is modeled by the function

$$\boxed{Q(t) = Q_0 e^{-rt}}$$

where $r = \frac{\ln(2)}{t_0}$.

Note: If t_0 denotes the half-life of a radioactive substance, we can rewrite the expression for $Q(t)$ as follows

$$Q(t) = Q_0\, e^{-rt} = Q_0\, e^{-(\ln(2)/t_0)\cdot t} = Q_0 \left(e^{\ln(2)}\right)^{-t/t_0} = Q_0\, 2^{-t/t_0} = Q_0\,(2^{-1})^{t/t_0} = Q_0 \left(\frac{1}{2}\right)^{t/t_0}.$$

Example 18: The half-life of Cesium-137 is 30 years. Suppose we have a 100 gram sample. How much of the sample will remain after 50 years?

Recall, $r = \frac{\ln(2)}{t_0}$ ← half-life

So $r = \frac{\ln(2)}{30}$

$Q_0 = 100$

$Q(t) = 100e^{-\frac{\ln(2)}{30} t}$

$Q(50) = 100e^{-\frac{\ln(2)}{30} \cdot 50} \approx \boxed{31.498 \text{ grams}}$