MA123, Chapter 1: Equations, functions, and graphs (pp. 1-15, Gootman)

`Chapter Goals:`

- Solve an equation for one variable in terms of another
- Review of functions
- Graphs of functions
- Linear functions
- Piecewise defined functions.
- Properties of exponents
- Review of factoring

`Assignments:` `Assignment 01`

This chapter touches on some of the review material we will make use of in our study of calculus. If you want more examples or practice with this or other review materials, check out the links given under **Prerequisites** on the main tab of the course website.

▶ **Equations and solution(s) to equations:** Roughly speaking, an **equation** is a statement that two mathematical expressions are equal. For instance, $x^3 - 2xy + y^2 = 5$ is an equation relating x and y. A set of numbers that can be substituted for the variables in an equation so that the equality is true is a **solution** for the equation. A solution is said to **satisfy** the equation.

▶ **Equations into functions:** An equation in two (or more) variables can sometimes be solved in terms of one of the variables. This type of equation is closely related to the notion of a **function**.

Example 1: Solve the equation $x^3 + 2xy + 5y = 7$ for y in terms of x.

$x^3 + 2xy + 5y = 7$ (Subtract x^3)

$2xy + 5y = 7 - x^3$ (Factor y)

$y(2x+5) = 7 - x^3$ (Divide by $2x+5$)

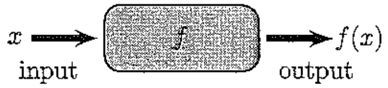
$\dfrac{y\cancel{(2x+5)}}{\cancel{2x+5}} = \dfrac{7-x^3}{2x+5}$ (Cancel)

$y = \dfrac{7-x^3}{2x+5}$

Observe that in the equation $y = \dfrac{7-x^3}{2x+5}$, the expression on the right-hand side can be viewed as a recipe that associates to any given value of x precisely one corresponding value for y.

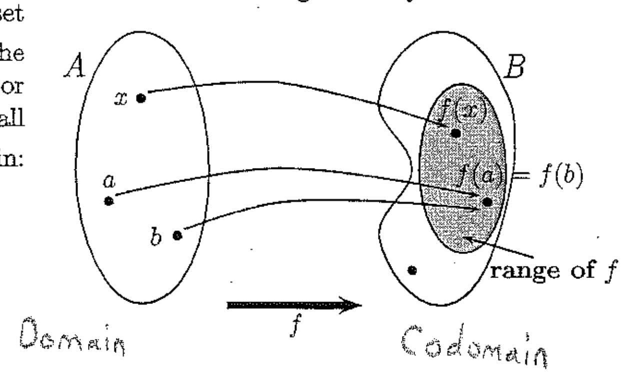
Definition of function:

A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B. The set A is called the **domain** of f whereas the set B is called the **codomain** of f; $f(x)$ is called the **value of f at x**, or the **image of x under f**. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain: range of $f = \{f(x) \,|\, x \in A\}$.



Machine diagram of f

Arrow diagram of f



Evaluating a function: The symbol that represents an arbitrary number in the domain of a function f is called an **independent variable**. The symbol that represents a number in the range of f is called a **dependent variable**. In the definition of a function the independent variable plays the role of a "placeholder." For example, the function $f(x) = 2x^2 - 3x + 1$ can be thought of as

$$f(\square) = 2 \cdot \square^2 - 3 \cdot \square + 1.$$

To evaluate f at a number (expression), we substitute the number (expression) for the placeholder.

Note: If f is a function of x, then $y = f(x)$ is a special kind of equation, in which the variable y appears alone on the left side of the equal sign and the expression on the right side of the equal sign involves only the other variable x. **Conversely**, when we have this special kind of equation, such as $y = e^x + x^3 - 3x + 5$, it is common to think of the right hand side as defining a function $f(x)$, and of the equation as being simply $y = f(x)$.

Example 2: Find the domain of the following functions:

$f(x) = \sqrt{3-x}$

Set $3 - x \geq 0$ (Add x)

$3 - x + x \geq 0 + x$ (Simplify)

$3 \geq x$

3

Interval notation: $(-\infty, 3]$

Set notation: $\{x \in \mathbb{R} : x \leq 3\}$

$g(x) = \dfrac{1}{x^2 - 4}$

Denom., $x^2 - 4$ can't be 0

$x^2 - 4 = 0$ (Diff. of squares)

$(x-2)(x+2) = 0$

$x - 2 = 0$, $x + 2 = 0$

$x = 2$, $x = -2$

−2 2

Interval notation: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

Set notation: $\{x \in \mathbb{R} : x \neq -2, 2\}$

$h(x) = \dfrac{1}{x} + \sqrt{x+2}$

Can't be 0

$x \neq 0$

AND

$x + 2 \geq 0$ (Subtract 2)

$x + 2 - 2 \geq 0 - 2$ (Simplify)

$x \geq -2$

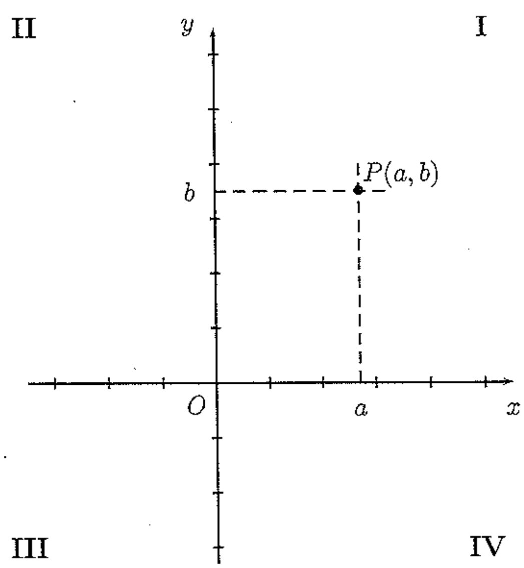
−2 0

Interval notation: $[-2, 0) \cup (0, \infty)$

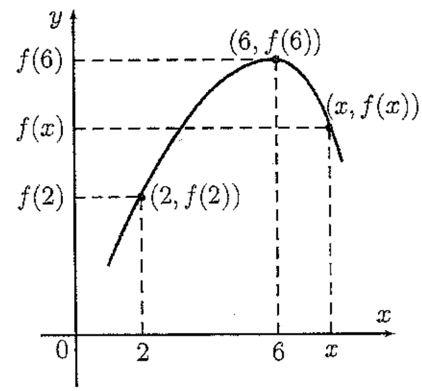
Set notation: $\{x \in \mathbb{R} : x \geq -2, x \neq 0\}$

► **Cartesian plane and the graph of a function:**

Points in a plane can be identified with ordered pairs of numbers to form the *coordinate plane.* To do this, we draw two perpendicular oriented lines (one horizontal and the other vertical) that intersect at 0 on each line. The horizontal line with positive direction to the right is called the *x-axis*; the other line with positive direction upward is called the *y-axis*. The point of intersection of the two axes is the *origin O*. The two axes divide the plane into four *quadrants*, labeled I, II, III, and IV. The coordinate plane is also called *Cartesian plane* in honor of the French mathematician/philosopher René Descartes (1596-1650). Any point P in the coordinate plane can be located by a unique ordered pair of numbers (a, b) as shown in the picture. The first number a is called the *x-coordinate* of P; the second number b is called the *y-coordinate* of P.



Graphing functions:

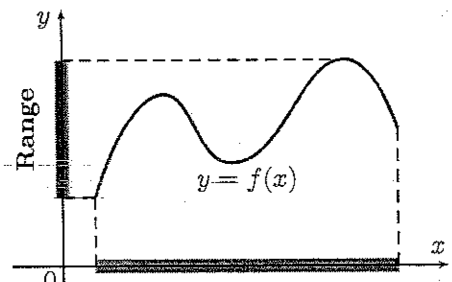


If f is a function with domain A, then the graph of f is the set of ordered pairs

$$\textbf{graph of } f = \{(x, f(x)) \,|\, x \in A\}.$$

In other words, the graph of f is the set of all points (x, y) such that $y = f(x)$; that is, the graph of f is the graph of the equation $y = f(x)$.

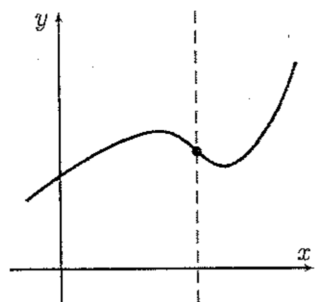
Obtaining information from the graph of a function:



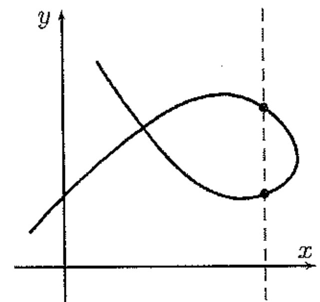
The values of a function are represented by the height of its graph above the x-axis. So, we can read off the values of a function from its graph.
In addition, the graph of a function helps us picture the domain and range of the function on the x-axis and y-axis as shown in the picture:

The graph of a function is a curve in the xy-plane. But the question arises: Which curves in the xy-plane are graphs of functions?

The vertical line test:
A curve in the coordinate plane is the graph of a function if and only if no vertical line intersects the curve more than once.



Graph of a function



Not a graph of a function

Example 3: The graph of the function $y = f(x)$ is given below. Assume the entire graph is shown.

(a) What is the value of $f(0)$?

This is the y-coordinate when $x = 0$

$\Rightarrow y = \boxed{-2}$

(b) For what value(s) of x is $\underbrace{f(x) = 0}_{\text{when } y = 0}$?

This occurs when $\boxed{x = -1, 4}$

(c) What is the domain of $f(x)$?

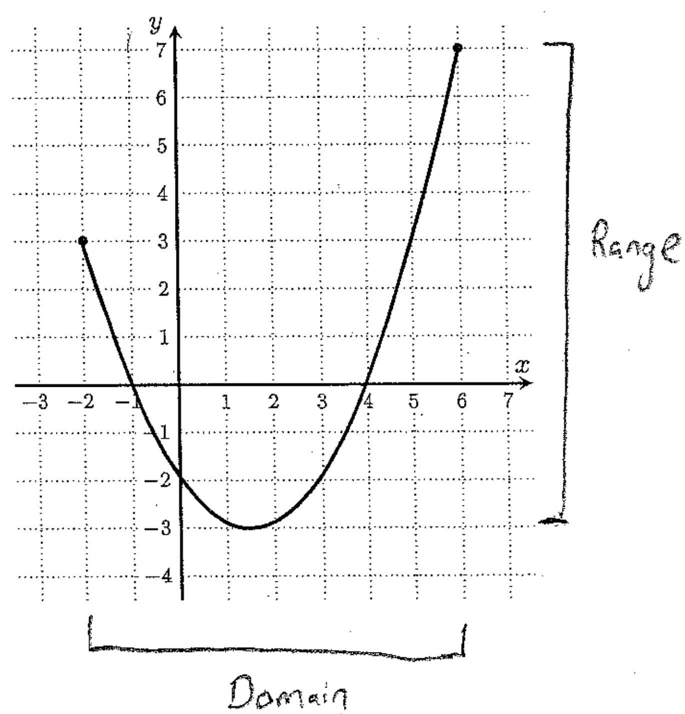
$\boxed{\text{Domain: } [-2, 6]}$

= All possible x-values

(d) What is the range of $f(x)$?

$\boxed{\text{Range: } [-3, 7]}$

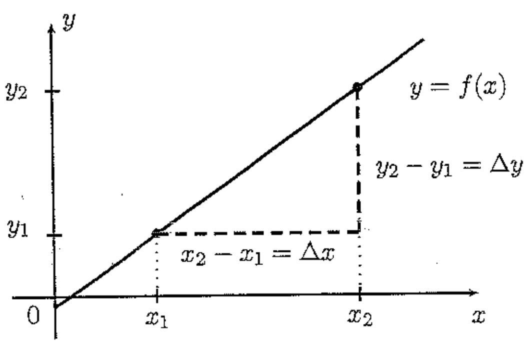
= All possible y-values



▶ **Lines and Linear Functions:** A **linear function** is a function whose graph is a straight line.

Slope of a Line: The slope of a (non-vertical) line can be determined by any two distinct points, (x_1, y_1) and (x_2, y_2), on the line:

$$\text{Slope} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x}.$$



Point-Slope Form of a Line:

If a line has slope m and passes through the point (x_0, y_0) and (x, y) is another point on the line, then

$$m = \frac{y - y_0}{x - x_0} \implies y - y_0 = m \cdot (x - x_0)$$

Slope-Intercept Form of a Line:

A linear function can be written in the form

$$f(x) = y = mx + b,$$

where m is the slope and b is the y-**intercept**.

You will need to be comfortable using both the point-slope and the slope-intercept forms of lines.

Example 4: Suppose a line passes through the points $(3, 4)$ and $(-1, 6)$. Write the equation of the line

(a) in point-slope form,

$y - y_0 = m(x - x_0)$

(x_0, y_0) = point on graph

We could take either $(3,4)$ or $(-1,6)$, so let's just take $(3,4)$

$m = \frac{\Delta y}{\Delta x} = \frac{6-4}{-1-3} = \frac{2}{-4} = -\frac{1}{2}$

equation: $\boxed{y - 4 = -\frac{1}{2}(x-3)}$

Note: $y - 6 = -\frac{1}{2}(x+1)$ is another possible answer

(b) and in slope intercept form.

Rearrange point-slope form to isolate y

$y - 4 = -\frac{1}{2}(x-3)$ (Distribute)

$y - 4 = -\frac{1}{2}x + \frac{3}{2}$ (Add 4)

$y = -\frac{1}{2}x + \frac{3}{2} + 4$ (Simplify)

$\boxed{y = -\frac{1}{2}x + \frac{11}{2}}$

Example 5: Suppose the linear function f satisfies $f(1.5) = 2$ and $f(3) = 5$. Determine $f(4)$.

Find slope and point

point: $(3, 5)$ (or $(1.5, 2)$)

slope: $m = \frac{5-2}{3-1.5} = \frac{3}{1.5} = 2$

equation: $y - y_0 = m(x - x_0)$

$y - 5 = 2(x - 3)$

Plug in $x = 4$:

$y - 5 = 2(4-3)$

$y - 5 = 2(1) = 2$

$y = 2 + 5$

$y = 7$

$\boxed{f(4) = 7}$

▶ **Parabolas and Quadratic Functions:**

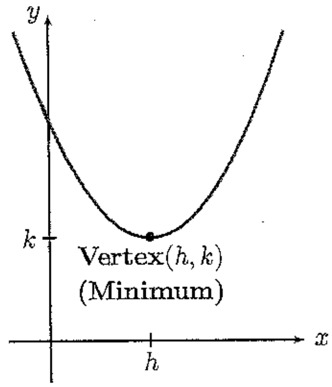
A **quadratic function** is a function f of the form

$$f(x) = ax^2 + bx + c,$$

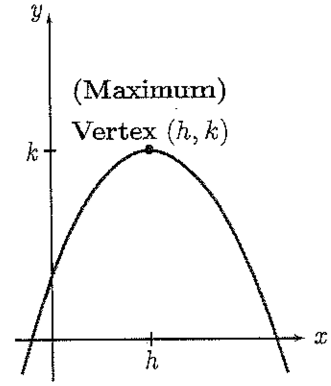
where a, b, and c are real numbers and $a \neq 0$.
The graph of any quadratic function is a parabola; it can be obtained from the graph of $f(x) = x^2$ by using shifting, reflecting, and stretching transformations.
Indeed, by completing the square a quadratic function $f(x) = ax^2 + bx + c$ can be expressed in the **standard form** or **vertex form**

$$f(x) = a(x-h)^2 + k.$$

The graph of f is a parabola with vertex (h, k); the parabola opens upward if $a > 0$, or downward if $a < 0$.



$f(x) = a(x-h)^2 + k,$
$a > 0$



$f(x) = a(x-h)^2 + k,$
$a < 0$

▶ **Piecewise Defined Functions:** A **piecewise defined function** is a function given by several different rules. In order to evaluate a piecewise defined function, you first need to decide which rule applies to the given value of the independent variable.

Example 6: Suppose

$$f(x) = \begin{cases} 2x+1, & \text{for } x \leq -1; \\ x^2, & \text{for } -1 < x < 2; \\ -3, & \text{for } 2 \leq x. \end{cases}$$

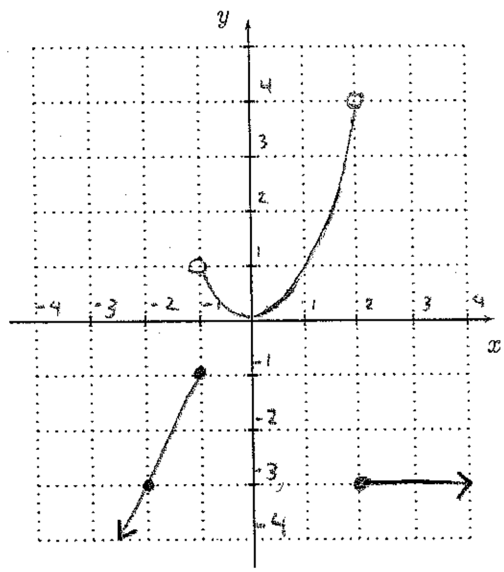
(a) Find each of $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, $f(3)$, $f(4)$

$f(-2) = 2(-2)+1 = -4+1 = -3$ $\quad f(1) = (1)^2 = 1$

$f(-1) = 2(-1)+1 = -2+1 = -1$ $\quad f(3) = -3$

$f(0) = (0)^2 = 0$ $\quad f(4) = -3$

(b) Sketch the graph of $f(x)$.



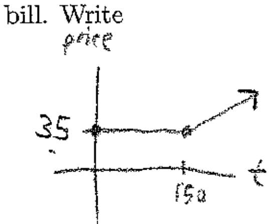
Example 7: Sam's cell phone provider charges him \$35.00 per month for basic service, which includes 150 "anytime minutes." Sam is charged an extra \$0.75 for each minute beyond 150 minutes. Let t denote the number of minutes that Sam used in a given month and $B(t)$ denote the amount of Sam's cell phone bill. Write a piecewise defined function for $B(t)$.

$$B(t) = \begin{cases} 35 & t \leq 150 \\ 35 + .75(t-150) & t > 150 \end{cases}$$

base price

rate each minute

only charge for minutes in excess of 150



Example 8 (Greatest Integer Function):

The **greatest integer function** (a.k.a, **unit step function**), denoted $[\![x]\!]$, associates to each real number x the greatest integer less than or equal to x.

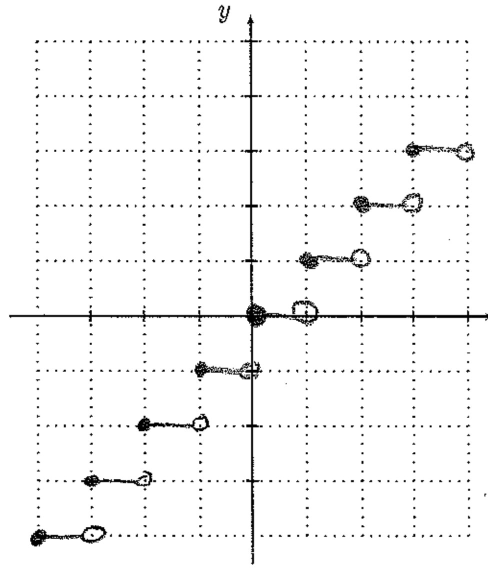
(a) Find each of $[\![0]\!]$, $[\![1]\!]$, $[\![2]\!]$, $[\![1.2]\!]$, $[\![1.97]\!]$, $[\![-1.8]\!]$

$[\![0]\!]=0$ $\quad [\![1.2]\!]=1$

$[\![1]\!]=1$ $\quad [\![1.97]\!]=1$

$[\![2]\!]=2$ $\quad [\![-1.8]\!]=-2$

(b) Sketch the graph of $f(x)$.



The greatest integer function will appear several times throughout the course. You may want to commit its graph to memory.

► **Exponential Notation:**

If a is <u>any</u> real number and n is a <u>positive</u> integer, then the **n-th power** of a is

$$a^n = \underbrace{a \cdot a \cdot \ \ldots \ \cdot a}_{n \text{ times}}$$

The number a is called the **base** whereas n is called the **exponent**.

The first and second laws of exponents below allow us to define a^n for any integer n.

Now, we want to define, for instance, $a^{1/3}$ in a way that is consistent with the laws of exponents. We would like:

$$\left(a^{1/3}\right)^3 = a^{(1/3)3} = a^1 = a; \qquad \text{thus} \qquad a^{1/3} = \sqrt[3]{a}$$

So, by the definition of nth root, we have:

$$\boxed{a^{1/n} = \sqrt[n]{a}}$$

Definition of rational exponents: For any rational exponent m/n in lowest terms, where m and n are integers and $n > 0$, we define

$$a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m \qquad \text{or equivalently}$$

$$a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}$$

If n is even we require that $a \geq 0$.

In the table below, a and b are real numbers ($\neq 0$ if needed) and the exponents x and y are rational numbers.

Laws of exponents:

(1.) $a^0 = 1$

(2.) $a^{-x} = \dfrac{1}{a^x}$

(3.) $a^x a^y = a^{x+y}$

(4.) $\dfrac{a^x}{a^y} = a^{x-y}$

(5.) $(a^x)^y = a^{xy}$

(6.) $(ab)^x = a^x b^x$

(7.) $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

For examples 9–11, assume all variables represent nonzero real numbers.

Example 9: Simplify the expression $\frac{x^6(3x)^4}{x^{24}}$.

$$\frac{x^6(3x)^4}{x^{24}}$$

$$= \frac{x^6 \cdot 3^4 x^4}{x^{24}}$$

$$= \frac{3^4 \cdot x^{10}}{x^{24}} = \frac{3^4}{x^{14}} = \frac{81}{x^{14}} \text{ or } 81x^{-14}$$

either of these might be considered "simplified," depending on our goal

Example 10: Simplify the expression $\frac{42z^{14} - 72z^9 + 24z^5}{-6z^4}$.

$$= \frac{42z^{14}}{-6z^4} + \frac{-72z^9}{-6z^4} + \frac{24z^5}{-6z^4}$$

Split into Three fractions

$$= -7z^{10} + 12z^5 - 4z$$

simplify each fraction

Example 11: Simplify the rational expression $x^{-3} + x^{-5}$ in the form $\frac{A}{B}$. Express the final result in a single fraction using positive exponents only.

$$x^{-3} + x^{-5}$$

$$= \frac{1}{x^3} + \frac{1}{x^5}$$

(get Common denom.)

$$= \frac{1}{x^3}\left(\frac{x^2}{x^2}\right) + \frac{1}{x^5}$$

$$= \frac{x^2}{x^5} + \frac{1}{x^5}$$

$$= \frac{x^2+1}{x^5}$$

▶ **Factoring:** Another skill we will find useful is factoring. The following examples will help us review.

Example 12: Simplify the rational expression $\frac{x^2 - x - 12}{x^2 + x - 20}$.

$$\frac{x^2 - x - 12}{x^2 + x - 20} \qquad \text{(factor)}$$

$$= \frac{(x-4)(x+3)}{(x+5)(x-4)} \qquad \text{(cancel)}$$

$$= \frac{x+3}{x+5} \quad \text{if } x \neq 4 \longleftarrow \text{(if } x=4\text{, the denom. equals 0 which makes the fraction undefined)}$$

Example 13: Simplify the rational expression $\frac{6x - x^2}{x^2 - 36}$.

$$\frac{6x - x^2}{x^2 - 36} \qquad \text{(factor)}$$

$$= \frac{x(6-x)}{(x-6)(x+6)} \qquad \text{(factor } -1 \text{ out)}$$

$$= \frac{-x\cancel{(-6+x)}}{\cancel{(x-6)}(x+6)} \qquad \text{(cancel, for } x \neq 6\text{)}$$

$$= \frac{-x}{x+6}$$