MA123, Chapter 10: Idea of the Integral

Chapter Goals:

- Understand the relationship between the area under a curve and the definite integral.
- Understand the relationship between velocity (speed), distance and the definite integral.
- Use the definite integral to compute the average value of a function over an interval

Assignments: Assignment 18 Assignment 19

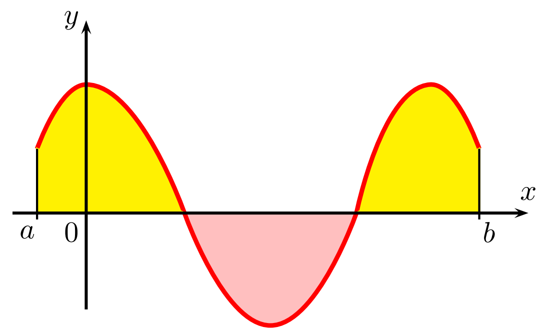
▶ **The basic idea:** Definite integrals compute *signed area.*

Definition: The **definite integral**

$$\int_a^b f(x)\,dx$$

computes the signed area between the graph of $y = f(x)$ and the x-axis on the interval $[a, b]$.

- If $a < b$ and the region is above the x-axis, the area has positive sign.
- If $a < b$ and the region is below the x-axis, the area has negative sign.
- If the function takes on both positive and negative values on $[a, b]$, the "positive" and "negative" areas will cancel out.



That is, if $a < b$, then

$$\int_a^b f(x)\,dx = [\text{area of the region(s) lying above the } x\text{-axis}] - [\text{area of the region(s) lying below the } x\text{-axis}]$$

Notation: Given $\int_a^b f(x)\,dx$, we call $f(x)$ the **integrand**, dx identifies x as the variable, and a and b are called the **limits of integration**.

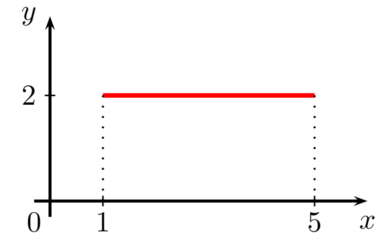
Applications: Suppose $v(t)$ measures the velocity of an object at time t.

(a) $\int_a^b v(t)\,dt$ measures the displacement of the object from $t = a$ to $t = b$. The displacement is the difference between the object's ending point and starting point.

(b) $\int_a^b |v(t)|\,dt$ measures the total distance traveled between $t = a$ and $t = b$.

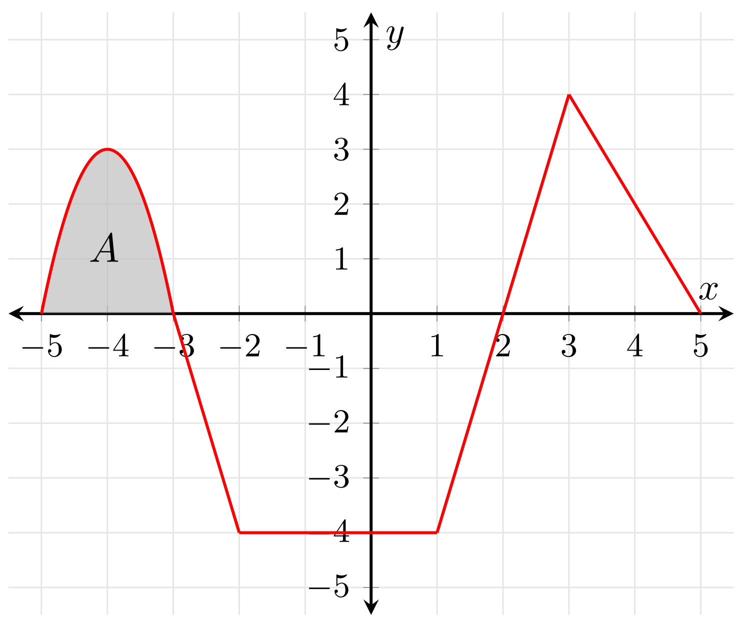
If $v(t)$ is always positive, *displacement* and *distance traveled* are the same.

Example 1 (Easy area problem): Find the area of the region in the xy-plane bounded above by the graph of the function $f(x) = 2$, below by the x-axis, on the left by the line $x = 1$, and on the right by the line $x = 5$.



Example 2 (Easy distance traveled problem): Suppose a car is traveling due east at a constant velocity of 55 miles per hour. How far does the car travel between noon and 2:00 pm?

Example 3: Use the graph of $f(x)$ shown to find the following integrals, given that the shaded region has area A.



(a) $\int_{2}^{5} f(x)\,dx$

(b) $\int_{-3}^{2} f(x)\,dx$

(c) $\int_{-3}^{5} f(x)\,dx$

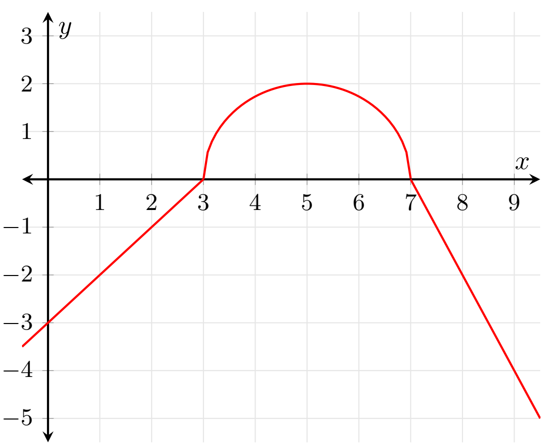
(d) $\int_{-5}^{5} f(x)\,dx$

Example 4: Suppose $f(x)$ is the greatest integer function, i.e., $f(x)$ equals the greatest integer less than or equal to x. So for example $f(2.3) = 2$, $f(4) = 4$, and $f(6.9) = 6$.

Find $\int_6^{10} f(x)\,dx$.

Example 5: Consider $g(x)$ shown here. The graph from $x = 3$ to $x = 7$ is a semicircle.

Find $\int_1^9 g(x)\,dx$.



Example 6: Let

$$f(x) = \begin{cases} 0 & \text{if } x < -5 \\ 4 & \text{if } -5 \le x < 0 \\ -3 & \text{if } 0 \le x < 2 \\ 0 & \text{if } x \ge 2 \end{cases}$$

and $g(x) = \int_{-5}^{x} f(t)\, dt$.

Determine the value of each of the following:

(a) $g(-10)$

(b) $g(-1)$

(c) $g(1)$

(d) $g(6)$

(e) What is the absolute maximum of $g(x)$?

► **Some properties of definite integrals:**

1. $\displaystyle\int_a^a f(x)\,dx = 0$

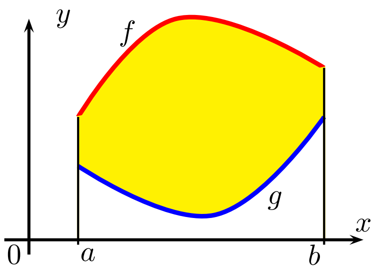
2. $\displaystyle\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

3. $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \left(\int_a^b f(x)\,dx\right) \pm \left(\int_a^b g(x)\,dx\right)$

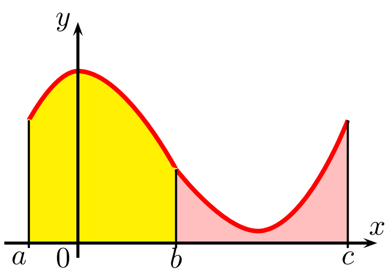
4. $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

5. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Geometric illustration of some of the above properties:



Property **3.** says that if f and g are positive-valued functions with f greater than g, then $\displaystyle\int_a^b (f(x) - g(x))\,dx$ gives the area between the graphs of f and g. However, we can rephrase this as the area under g subtracted from the area under f, which is given by $\displaystyle\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$.



Property **4.** says that if $f(x)$ is a positive valued function, then the area underneath the graph of $f(x)$ between a and b plus the area underneath the graph of $f(x)$ between b and c equals the area underneath the graph of $f(x)$ between a and c.

Property **5.** follows from Properties **4.** and **1.** by letting $c = a$.

Example 7: Using the graph of $f(x)$ from Example 3, find the integral $\displaystyle\int_2^5 5f(x)\,dx$.

Example 8: Let

$$\int_1^4 f(x)\,dx = 3, \qquad \int_1^9 f(x)\,dx = -4, \qquad \int_1^4 g(x)\,dx = 2, \qquad \int_1^9 g(x)\,dx = 8, \qquad \int_6^9 g(x)\,dx = 3.$$

Use these values to evaluate the given definite integrals.

(a) $\displaystyle\int_1^4 (f(x) - g(x))\,dx$

(b) $\displaystyle\int_9^1 (f(x) + g(x))\,dx$

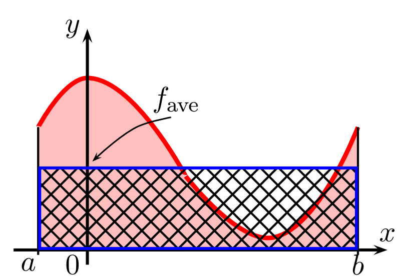
(c) $\displaystyle\int_4^9 (7f(x) + 10g(x))\,dx$

(d) $\displaystyle\int_4^6 (g(x) - 5)\,dx$

► **Average Values:** The average of finitely many numbers $y_1, y_2, \ldots, y_n$ is $y_{\text{ave}} = \dfrac{y_1 + y_2 + \cdots + y_n}{n}$. What if we are dealing with infinitely many values? More generally, how can we compute the average of a function f defined on an interval?

Average of a function: The average of a function f on an interval $[a, b]$ equals the integral of f over the interval divided by the length of the interval:

$$f_{\text{ave}} = \frac{\displaystyle\int_a^b f(x)\,dx}{b - a}.$$



Geometric meaning: If f is a positive valued function, f_{ave} is that number such that the rectangle with base $[a, b]$ and height f_{ave} has the same area as the region underneath the graph of f from a to b.

Example 9: Suppose $f(x) = \begin{cases} 2 & \text{if } x \leq 5 \\ \frac{1}{2}(5x - 21) & \text{if } x > 5. \end{cases}$

Find the average value of $f(x)$ over the interval $[2, 7]$.

Example 10: Suppose $f(x) = \begin{cases} -3 & \text{if } 4 \leq x < 7 \\ 5 & \text{if } 7 \leq x < 9. \end{cases}$

(a) Find the average value of $f(x)$ on the interval $[4, 9]$.

(b) Find the average rate of change of $f(x)$ on the interval $[4, 9]$