MA123, Chapter 3: The idea of limits

Chapter Goals:	• Evaluate limits. • Evaluate one-sided limits. • Understand continuity.

Assignments: Assignment 04 Assignment 05

Earlier, the idea of limits came up naturally in the course of defining the derivative of a function at a point. We now study limits more systematically. *Computing a limit means computing what happens to the value of a function as the variable in the expression gets closer and closer to (but does not equal) a particular value.*

▶ **The basic definition of limit:** Let f be a function of x. The expression

$$\lim_{x \to c} f(x) = L$$

means that as x gets closer and closer to c, through values both smaller and larger than c, but not equal to c, then the values of $f(x)$ get closer and closer to the value L.

Note: It may sometimes happen that the limit does not exist.

Example 1 (a): Use the tables to help evaluate $\lim_{x \to 2} \dfrac{x^2+8}{x+2}$.

x gets close to 2 from the left				
x	1.8	1.9	1.99	1.999
$\frac{x^2+8}{x+2}$	2.9578	2.9769...	2.9975...	2.9997...

x gets close to 2 from the right				
2.001	2.01	2.1	2.2	x
3.0002...	3.0025...	3.0268...	3.0571...	$\frac{x^2+8}{x+2}$

Example 1 (b): Suppose that, instead of calculating all the values in the above tables, you simply substitute the value $x = 2$ into $\dfrac{x^2+8}{x+2}$. What do you find? $\dfrac{2^2+8}{2+2} = \dfrac{12}{4} = 3$

Note: The method of **substituting in** the limiting value of the variable works because the operations of arithmetic, namely, addition, subtraction, multiplication, and division, all behave reasonably with respect to the idea of 'getting closer to' as long as nothing illegal happens. The one illegality you will mainly have to watch out for is 'division by zero'. More precisely, if f and g are two functions one has:

$$\lim_{x \to c} \big(f(x) + g(x)\big) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) \qquad \lim_{x \to c} \big(f(x) - g(x)\big) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$$

$$\lim_{x \to c} \big(f(x) \cdot g(x)\big) = \left(\lim_{x \to c} f(x)\right) \cdot \left(\lim_{x \to c} g(x)\right) \qquad \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$$

as long as $\lim_{x \to c} g(x) \neq 0$

Example 2: Compute $\lim_{x\to 1} \left((x^2+4x+3)\cdot(2x-4)\right)$.

$$= \lim_{x\to 1}(x^2+4x+3) \cdot \lim_{x\to 1}(2x-4)$$

evaluate each expression at $x=1$

$$= \left(\lim_{x\to 1} x^2 + \lim_{x\to 1} 4x + \lim_{x\to 1} 3\right)\cdot\left(\lim_{x\to 1} 2x - \lim_{x\to 1} 4\right) = (1+4+3)(2-4)$$

$$= 8(-2) = \boxed{-16}$$

Example 3: Compute $\lim_{x\to 1} \dfrac{x^2-2x+1}{x+1}$.

$$= \frac{\lim_{x\to 1}(x^2-2x+1)}{\lim_{x\to 1}(x+1)} = \frac{1-2+1}{1+1} = \frac{0}{2} = \boxed{0}$$

denominator is not zero, so this is allowed

Example 4: Suppose $\lim_{x\to 3} f(x) = -2$ and $\lim_{x\to 3} g(x) = 4$. Determine

$$\lim_{x\to 3}\left((x+1)\cdot f(x)^2 + \frac{x+2}{g(x)}\right).$$

$$= \lim_{x\to 3}(x+1)\cdot\left(\lim_{x\to 3} f(x)\right)^2 + \frac{\lim_{x\to 3}(x+2)}{\lim_{x\to 3} g(x)} = (4)(-2)^2 + \frac{5}{4}$$

$$= 16 + \frac{5}{4} = \boxed{\frac{69}{4}}$$

► **Some complications with the definition of limits:** The previous examples seem to imply that "computing a limit" is the same thing as "evaluating a function". This is only true if the function in the limit is "nice enough" ("nice enough" will be defined more precisely in a few pages).

The next few examples will illustrate that the computation of $\lim_{x\to c} f(x)$ does not always reduce to the mere substitution of the value of c in place of x in the expression defining $f(x)$. The 'unusual' functions described in what follows are introduced to emphasize the fact that the notion of limit really involves what happens to the values of $f(x)$ as x gets closer to the fixed value c, and not what the value of $f(x)$ at $x=c$ is. In addition, the most interesting limits generally arise precisely when substitution gives an illegal expression involving division by 0, or even an expression of the form $\frac{0}{0}$. The latter case occurs for example when computing the derivative of a function.

► **How can a limit fail to exist?** There are two basic ways that a limit can fail to exist.

(a) The function attempts to approach multiple values as $x \to c$.
Geometrically, this behavior can be seen as a jump in the graph of a function.
Algebraically, this behavior typically arises with piecewise defined funtions.

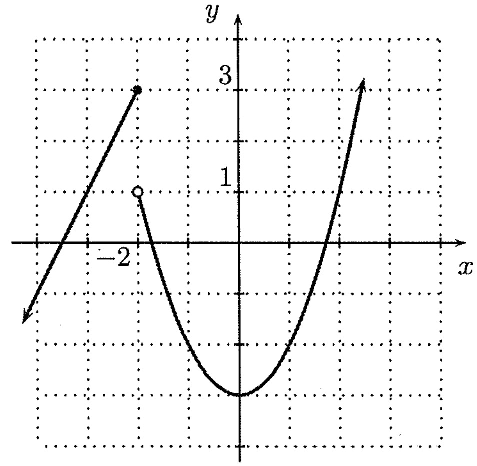
(b) The function grows without bound as $x \to c$.
Geometrically, this behavior can be seen as a vertical asymptote in the graph of a function.
Algebraically, this behavior typically arises when the denominator of a function approaches zero.

Example 5:

The graph of the function

$$h(x) = \begin{cases} x^2 - 3, & \text{if } x > -2; \\ 2x + 7, & \text{if } x \leq -2 \end{cases}$$

is shown to the right.



Analyze $\lim_{x \to -2} h(x)$. From the left, approaches $y = 3$.

From the right, approaches $y = 1$.

Thus $\lim_{x \to -2} h(x)$ $\boxed{\text{does not exist}}$

The previous example showed that the limit of $h(x)$ as the variable approached -2 did not exist. On the other hand, the function appears to have well defined limiting behavior on either side of $x = -2$. This brings us to the following notions:

One-sided limits: A one-sided limit expresses what happens to the values of an expression as the variable in the expression gets closer and closer to some particular value c from either the left on the number line (that is, through values less than c) or from the right on the number line (that is, through values greater than c). The notation is:

$$\underbrace{\lim_{x \to c^-} f(x)}_{\text{limit from the left of c}} \qquad \underbrace{\lim_{x \to c^+} f(x)}_{\text{limit from the right of c}}$$

Fact: $\lim_{x \to c} f(x)$ exists if and only if both $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist and have the same value.

Example 6:

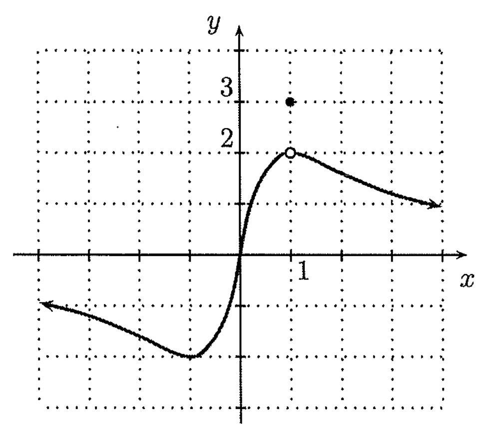
for all x-values close to 1 (but not $= 1$), $g(x) = \frac{4x}{x^2+2}$.

The graph of the function

$$g(x) = \begin{cases} \dfrac{4x}{x^2 + 1}, & \text{if } x \neq 1; \\ 3, & \text{if } x = 1. \end{cases}$$

is shown to the right.

x	$g(x)$
0.8	1.9512195
0.9	1.9889503
0.999	1.999999
1.001	1.999999
1.1	1.9909502
1.2	1.9672131

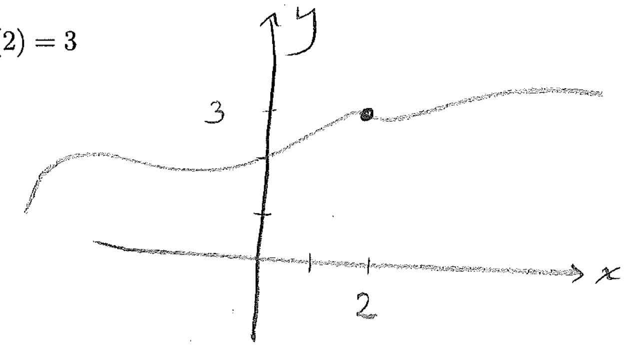


Compute $\lim_{x \to 1} g(x)$. $= \boxed{2}$

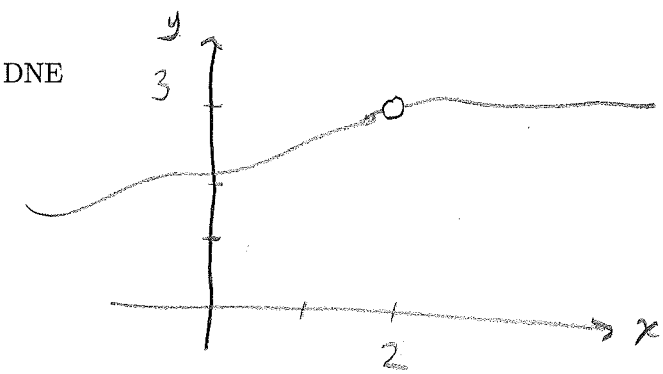
looks at x very close to (but not $=$) 1.

Example 7: In each case, sketch the graph of a function $y = f(x)$ satisfying the given properties.

(a) $\lim_{x \to 2} f(x) = 3, \quad f(2) = 3$

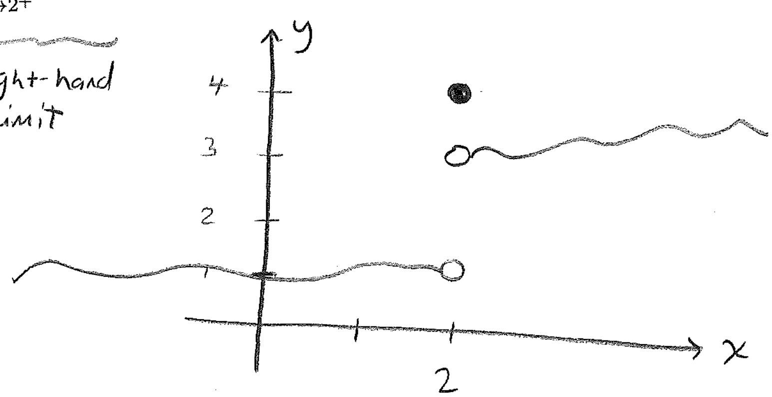


(b) $\lim_{x \to 2} f(x) = 3, \quad f(2)$ DNE



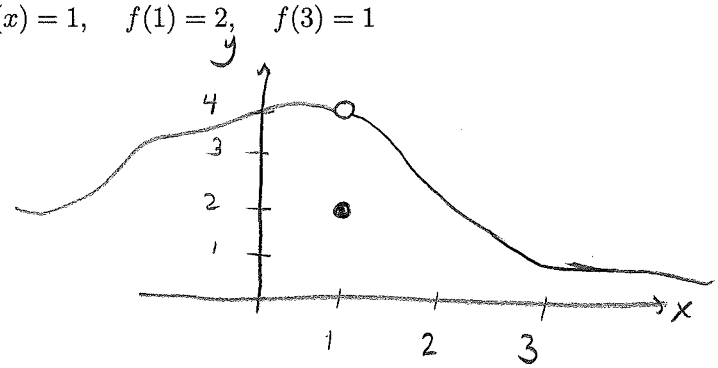
"open circle":
no y-value
at $x = 2$

(c) $\underbrace{\lim_{x \to 2^-} f(x) = 1}_{\text{left-hand limit}}, \quad \underbrace{\lim_{x \to 2^+} f(x) = 3}_{\text{right-hand limit}}, \quad f(2) = 4$



note that
$\lim_{x \to 2} f(x)$
DNE

(d) $\lim_{x \to 1} f(x) = 4, \quad \lim_{x \to 3} f(x) = 1, \quad f(1) = 2, \quad f(3) = 1$

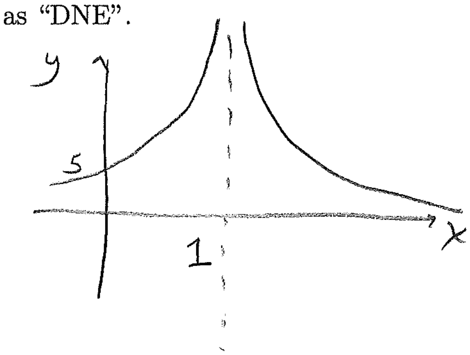


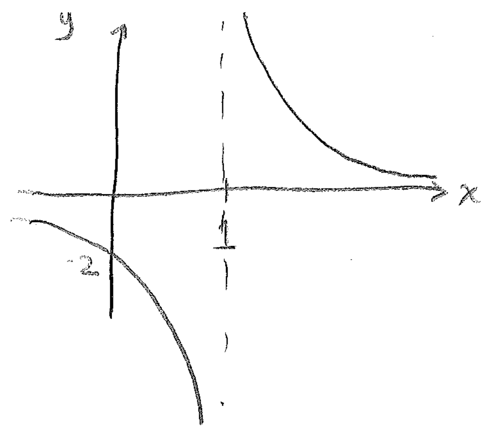
▶ **The problem of division by zero and a finite nonzero numerator:** When this happens, it is standard to say that the expression "is getting arbitrarily large (in the positive or negative direction)" or is "going to (positive or negative) infinity," denoted by $\pm\infty$. As infinity is not really a number, the expression is not really getting close to any particular real number. Thus, technically speaking, the limit does not exist. In the web homework system, "infinite limits" should be entered as "DNE".

Example 8: Analyze $\lim_{x\to 1} \frac{5}{(x-1)^2}$. $\boxed{\text{DNE}}$

test $x = 1$: $\frac{5}{0}$

approaches $+\infty$ from the left and the right of $x = 1$.

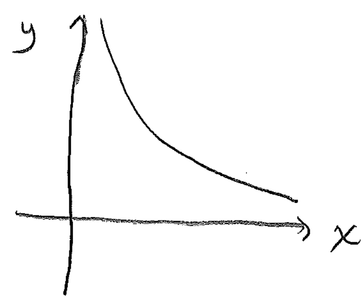




Example 9: Analyze $\lim_{x\to 1} \frac{2}{x-1}$. $\boxed{\text{DNE}}$

test $x = 1$: $\frac{2}{0}$

approaches $-\infty$ from the left, and $+\infty$ from the right



Example 10: Analyze the limit $\lim_{x\to 0} \frac{2}{\sqrt{x}}$

test $x = 0$: $\frac{2}{0}$

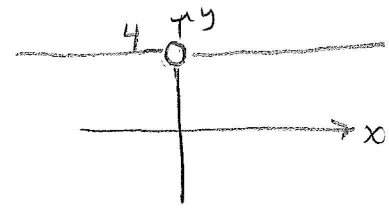
not defined for $x < 0$

approaches $+\infty$ from the right

Be sure to graph the functions in each of the last three examples, and notice the graphs have vertical asymptotes at $x = 1$, $x = 1$, and $x = 0$, respectively.

► $\boxed{\textbf{The case } \frac{0}{0}\textbf{:}}$ The most interesting and important situation with limits is when a substitution yields $\frac{0}{0}$. This is precisely the situation we are confronted with when attempting to compute derivatives from the definition. The result $\frac{0}{0}$ yields absolutely no information about the limit. It does not even tell us that the limit does not exist. The only thing it tells us is that we have to do more work to determine the limit.

Example 11: Find the limit $\lim_{x\to 0} \frac{4x}{x}$.



test $x = 0$: $\frac{0}{0}$ do more work!

$$\lim_{x\to 0} \frac{4x}{x} = \lim_{x\to 0} 4 = \boxed{4}$$

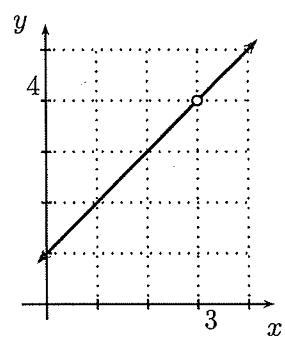
Example 12: Find the limit $\lim_{x\to 0} \left(\frac{2}{x} + \frac{5x-2}{x}\right)$. $= \lim_{x\to 0} \frac{2+5x-2}{x}$

$$= \lim_{x\to 0} \frac{5x}{x} = \lim_{x\to 0} 5 = \boxed{5}$$

test $x = 3$: $\frac{9-6-3}{3-3} = \frac{0}{0}$, do more work!

Example 13:

Find the limit $\lim_{x\to 3} \frac{x^2-2x-3}{x-3}$.



$$= \lim_{x\to 3} \frac{(x-3)(x+1)}{x-3} = \lim_{x\to 3} (x+1)$$

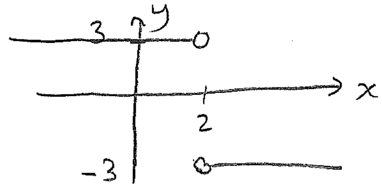
$$= 3+1 = \boxed{4}$$

Example 14: Find the limit $\lim_{h\to 0} \frac{(h-3)^2-9}{h}$. $= \lim_{h\to 0} \frac{h^2-6h+9-9}{h}$

$$= \lim_{h\to 0} \frac{h^2-6h}{h} = \lim_{h\to 0} \frac{h(h-6)}{h} = \lim_{h\to 0} (h-6) = 0-6$$

$$= \boxed{-6}$$

graph of $\frac{|3x-6|}{x-2}$ is



Example 15: Find the limits

$$\lim_{x\to 2^+} \frac{|3x-6|}{x-2} = 3$$

For $x > 2$, $3x-6$ is positive, so $|3x-6| = 3x-6$:

$$\frac{|3x-6|}{x-2} = \frac{3x-6}{x-2} = \frac{3(x-2)}{x-2}$$

$$\lim_{x\to 2^-} \frac{|3x-6|}{x-2} = \boxed{-3}$$

For $x < 2$, $3x-6$ is negative, so $|3x-6| = -(3x-6)$

$$\frac{|3x-6|}{x-2} = \frac{-(3x-6)}{x-2} = \frac{-3(x-2)}{x-2}$$

$$\lim_{x\to 2} \frac{|3x-6|}{x-2}. \boxed{\text{DNE}}$$

left and right limits don't match, so $\lim_{x\to 2} \frac{|3x-6|}{x-2}$ doesn't exist.

▶ **Limits at infinity:** A function $f(x)$ is said to be a *rational function* if it is of the type $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are both polynomials in x. Sometimes we are interested in determining the behavior of a rational function for large (positive or negative) values of the variable. This will be the case, for example, in Chapter 9.

There is a general principle that makes computing these limits easy. The **idea** is that, for very large (positive or negative) values of x, the term with the highest power of x has the most influence on the behavior of the polynomial. In other words, when x is very large, the term with the highest power dominates the other terms.

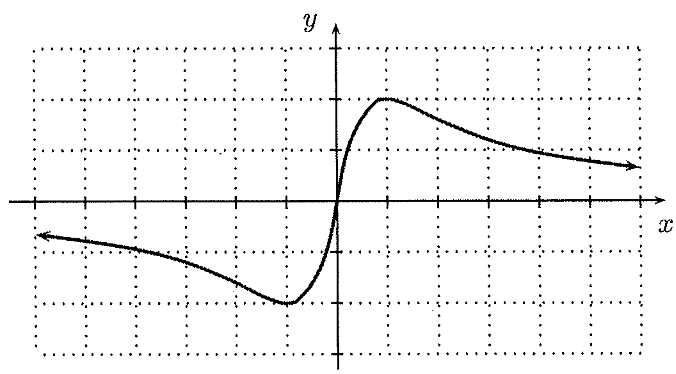
Theorem: Let $p(x)$ and $q(x)$ be polynomials. Then $\displaystyle\lim_{x\to\pm\infty} \frac{p(x)}{q(x)} = \lim_{x\to\pm\infty} \frac{\text{highest order term of } p(x)}{\text{highest order term of } q(x)}$.

Example 16:

Let $p(x) = \frac{4x}{x^2+1}$. Find the limits

$$\lim_{x\to+\infty} \frac{4x}{x^2+1} = \boxed{0} \qquad \lim_{x\to-\infty} \frac{4x}{x^2+1} = \boxed{0}$$

$$= \lim_{x\to\infty} \frac{4x}{x^2} = \lim_{x\to\infty} \frac{4}{x} = \boxed{0}$$



Example 17: Find the limit $\displaystyle\lim_{x\to\infty} \frac{(2x\cancel{+1})^2}{5x^2+2x+1}$.

$$= \lim_{x\to\infty} \frac{(2x)^2}{5x^2} = \lim_{x\to\infty} \frac{4x^2}{5x^2}$$

$$= \lim_{x\to\infty} \frac{4}{5} = \boxed{\frac{4}{5}}$$

For $x \to \infty$, disregard all but the highest power terms in each factor.

Example 18: Find the limit $\displaystyle\lim_{x\to\infty} \frac{(3x+2)^2(5x+1)\sqrt{4x^6+1}}{(x+1)(2x+3)^2(4x+5)^3}$

$\sqrt{\ }$ is the same as $(\ )^{\frac{1}{2}}$

$$= \lim_{x\to\infty} \frac{(3x)^2(5x)\sqrt{4x^6}}{x(2x)^2(4x)^3} = \lim_{x\to\infty} \frac{9x^2(5x)(2x^3)}{x(4x^2)(64x^3)} = \lim_{x\to\infty} \frac{90x^6}{256x^6}$$

$$\boxed{= \frac{90}{256}} \quad \boxed{= \frac{45}{128}}$$

▶ **Continuity:** A function f is **continuous at a point** $x = c$ if

$$\boxed{\lim_{x\to c} f(x) = f(c).}$$

A function f is **continuous on an interval** if it is continuous at every point of that interval.

Note: Geometrically, this means that the graph of f has no holes, jumps, or gaps at any point in the domain of f. Thus you can draw the graph of f from one end of the interval to the other without lifting your pencil off the paper.
Analytically, this means the value of the function at $x = c$ can be recovered if one knows the values of $f(x)$ for near $x = c$. In other words, the values of a continuous function cannot change abruptly.

Fact: If f and g are continuous functions at c then

$$\underset{\text{constant}}{\underset{\uparrow}{k}}f(x), \quad f(x)+g(x), \quad f(x)\cdot g(x) \quad \text{and} \quad \frac{f(x)}{g(x)}, \text{ where } g(c) \neq 0, \qquad \text{are continuous at } c.$$

Examples: Many of the standard algebraic functions are continuous.

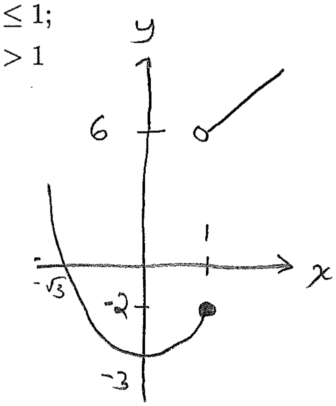
- Polynomials are continuous at every point.
- Rational functions are continuous at every point in their domain. (i.e., rational functions are continuous away from zeros of their denominators)

Example 19: Consider the function $f(x) = \begin{cases} x^2 - 3, & \text{if } x \leq 1; \\ 2x + B, & \text{if } x > 1 \end{cases}$

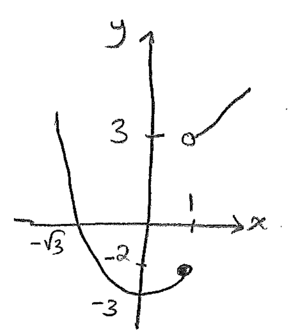
(a) Graph the function f when $B = 4$ and $B = -1$.

(b) Find a value of B such that the function is continuous at $x = 1$.

Case 1: $B = 4$ $\quad f(x) = \begin{cases} x^2 - 3 & \text{if } x \leq 1 \\ 2x + 4 & \text{if } x > 1 \end{cases}$



Case 2: $B = 1$ $\quad f(x) = \begin{cases} x^2 - 3 & \text{if } x \leq 1 \\ 2x + 1 & \text{if } x > 1 \end{cases}$



(b) For continuity, we must remove the "jump" at $x = 1$.
We need the left-hand limit to match the right-hand limit

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2 - 3) = 1^2 - 3 = -2$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x + B) = 2 + B$$

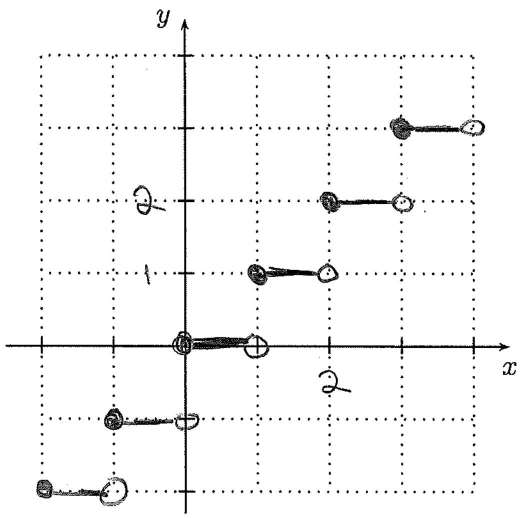
these must match

Set $2 + B = -2$

$$\boxed{B = -4}$$

also called the "floor function"
also notated $\lfloor x \rfloor$.

$\boxed{\textbf{Example 20:}}$ Let $f(x) = [\![x]\!]$ be the function that associates to any value of x the greatest integer less than or equal to x. Compute the $[\![0.5]\!]$, $[\![1.99]\!]$, $[\![2]\!]$, $[\![2.01]\!]$, $[\![4.87]\!]$, $[\![-1.5]\!]$.
Make a graph of the function $y = [\![x]\!]$. Compute $\lim\limits_{x\to 2^-} [\![x]\!]$ and $\lim\limits_{x\to 2^+} [\![x]\!]$.



$[\![0.5]\!] = 0$, $[\![1.99]\!] = 1$, $[\![2]\!] = 2$,
$[\![2.01]\!] = 2$, $[\![4.87]\!] = 4$, $\underbrace{[\![-1.5]\!] = -2}$

always chose the integer to the left on the number line

careful!

$\lim\limits_{x \to 2^-} [\![x]\!] = 1$

$\lim\limits_{x \to 2^+} [\![x]\!] = 2$

Look back at example 8 in Chapter 1 for more information about the greatest integer function.

$\boxed{\textbf{Example 21:}}$ Sketch the graph of a single function $y = f(x)$ satisfying all of the following conditions:

- $f(1) = 4$, $\iff$ The point $(1,4)$ is on the graph
- $\lim\limits_{x\to-\infty} f(x) = -2$,
- $\lim\limits_{x\to\infty} f(x) = 2$,
- $f(x)$ is continuous everywhere except at $x = -1$.

f will have a hole, jump, or vertical asymptote at $x=1$, but nowhere else

this example shows a jump; other graphs possible!

approaches $y = 2$ as $x \to +\infty$

approachs $y = -2$ as $x \to -\infty$