Chapter Goals:	• Understand the derivative as the slope of the tangent line at a point. • Understand differentiability. • Use the limit definition to calculate some derivatives.
Assignments:	Assignment 06 Assignment 07

In this chapter we explore further the relation between the derivative and the equation of the tangent line at a point. Then we learn how to compute the derivative of some functions using the definition of the derivative. One reason for doing this is to convince you that the rules and formulas for derivatives have a solid foundation and can be explained both geometrically and algebraically.

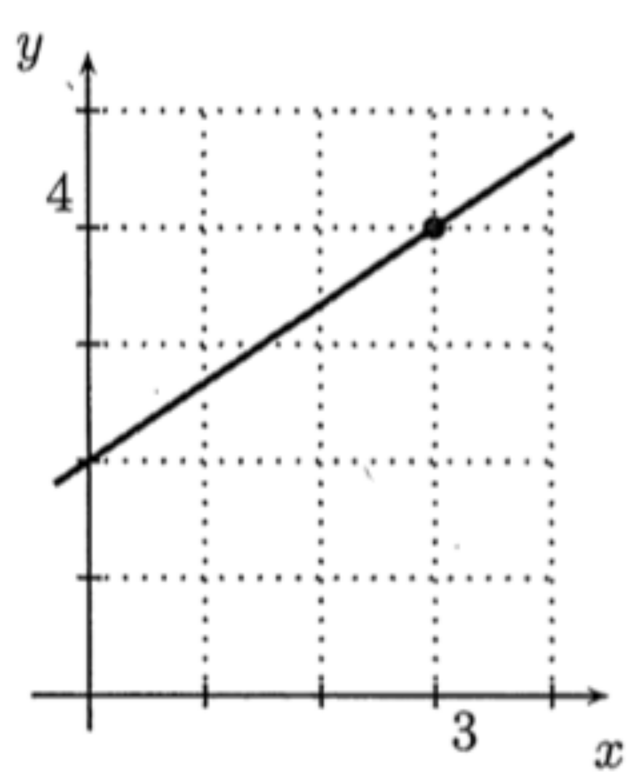
▶ **Tangent lines:** Recall from Chapter 2 the idea of the *tangent line* to the graph of a function. We can think of the tangent line at x_0 as a good approximation to the function for x-values near x_0.
For a given value x_0, *the derivative of f at x_0, namely $f'(x_0)$, gives the slope of the tangent line to the graph of f at the point* $(x_0, f(x_0))$. Thus, the equation of the tangent line to such a point is given by the formula

$$\boxed{y = f(x_0) + f'(x_0)(x - x_0).}$$

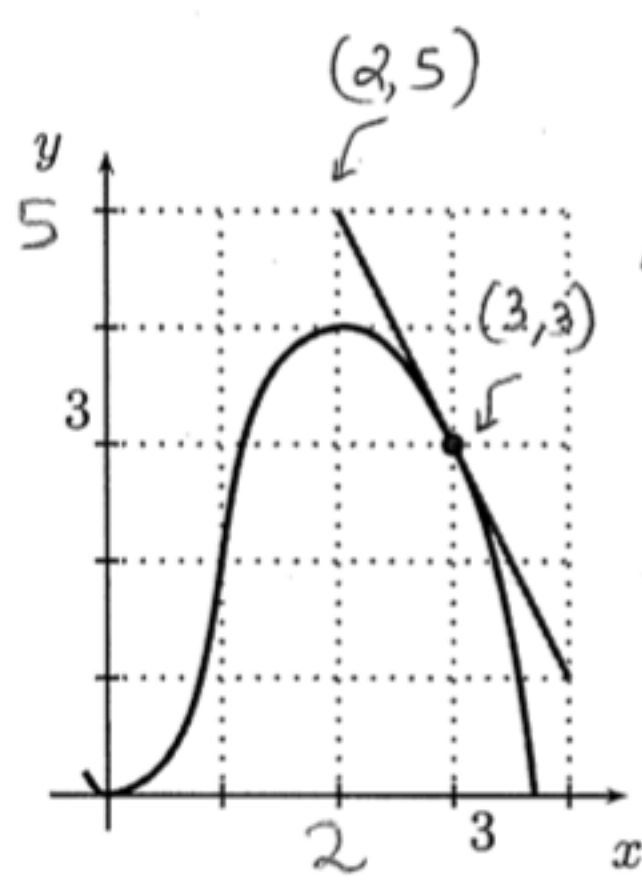
Example 1: The graph of a function $h(x)$ and the coordinates of a point $(x_0, h(x_0))$ on the graphs of h are given below. Find $h'(x_0)$ by analyzing the graph.

$h'(3)$ is the slope of the line, which goes through $(3,4)$ and $(0,2)$

$h'(3) = \frac{4-2}{3-0} = \boxed{\frac{2}{3}}$



$h'(3) \stackrel{?}{=}$



$h'(3) \stackrel{?}{=}$

$h'(3) \approx \frac{3-5}{3-2} = \frac{-2}{1} = \boxed{-2}$

Note: In the following problems you can use the fact that the derivative of $f(x) = ax^2 + bx + c$ is $f'(x) = 2ax + b$. (See the calculation carried out in Chapter 2, Example 15.)

Example 2: Consider the function $f(x) = 3x^2 - 6x - 10$. Write the equation of the tangent line to the graph of f at $x = -2$ in the form $y = mx + b$, for appropriate constants m and b.

y-value: $y = f(-2) = 3(-2)^2 - 6(-2) - 10 = 12 + 12 - 10 = 14$

$f'(x) = 2(3)x - 6 = 6x - 6$. At $x = -2$, $m = f'(-2) = 6(-2) - 6 = -18$

Line equation: $y - y_1 = m(x - x_1) \Rightarrow y - 14 = -18(x + 2)$

$y - 14 = -18x - 36 \Rightarrow \boxed{y = -18x - 22}$

add

using $f(x) = ax^2 + bx + c \Rightarrow f'(x) = 2ax + b$

Example 3: Consider the function $g(x) = -3x^2 + 7x - 6$. Write an equation of the tangent line to the graph of g at $x = 1$. For which values of y_1 and y_2 does this tangent line go through the points $(-1, y_1)$ and $(4, y_2)$?

y-value: $g(1) = -3(1)^2 + 7(1) - 6 = -3 + 7 - 6 = -2$

$g'(x) = 2(-3)x + 7 = -6x + 7$

slope $= g'(1) = -6(1) + 7 = 1.$

$y - y_1 = m(x - x_1) \Rightarrow y + 2 = 1(x - 1) \Rightarrow y + 2 = x - 1$

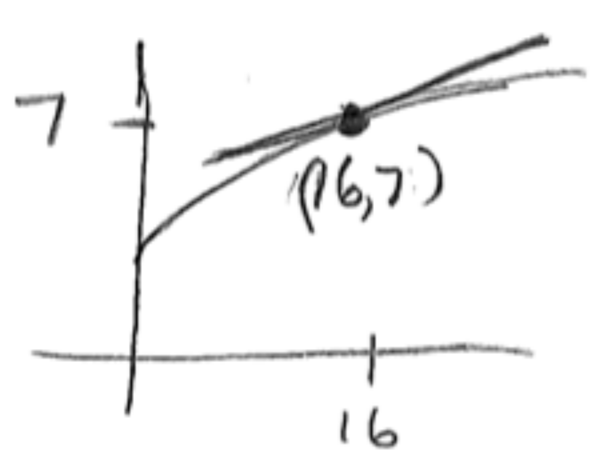
$\Rightarrow \boxed{y = x - 3}$ For the point $(-1, y_1)$, let $x = -1$:
$y = -1 - 3 = -4$ so $\boxed{y_1 = -4}$

For the point $(4, y_2)$, let $x = 4$: $y = 4 - 3$ so $\boxed{y_2 = 1}$

Example 4: Suppose that the equation of the tangent line to the graph of the function $f(x) = \sqrt{x} + a$ at $x = 16$ is given by $y = mx + 5$. Find a and m. (**Hint:** You may use $f'(x) = \frac{1}{2\sqrt{x}}$.)

m is the slope of the tangent line, so $\boxed{m} = f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{2(4)} = \boxed{\frac{1}{8}}$.

The tangent line is $y = \frac{1}{8}x + 5$. At $x = 16$, the y-value is $y = \frac{1}{8}(16) + 5 = 2 + 5 = 7$.



Thus, $f(x) = \sqrt{x} + a \Rightarrow f(16) = \sqrt{16} + a = 4 + a$

$\Rightarrow 4 + a = 7 \leftarrow$ y-value must also be 7

$\boxed{a = 3}$

▶ **Differentiability:** Differentiability is concerned with whether the graph of the function can be well approximated by a straight line. Geometrically, we say that $y = f(x)$ is differentiable at $x = c$ provided that the graph of $y = f(x)$ is (nearly) indistinguishable from the graph of a straight line, provided we restrict attention to x values that are sufficiently close to $x = c$.

More precisely, we say that f is **differentiable at** $x = c$ if the limit

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists. In other words, a function is differentiable at $x = c$ if the derivative exists at $x = c$.

Note: Geometrically, f is differentiable at x if there is a well defined tangent line to $y = f(x)$ at the point $(x, f(x))$ (so the graph is smooth there, and does not have a sharp point), and furthermore the tangent line is not vertical.
Analytically, a function is differentiable if the function does not abruptly change direction.

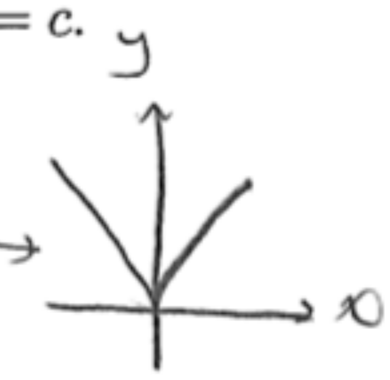
Examples: Many of the standard algebraic functions are differentiable.

- Polynomials are differentiable at every point.
- Rational functions are differentiable at every point in their domain. (i.e., rational functions are differentiable away from zeros of their denominators)

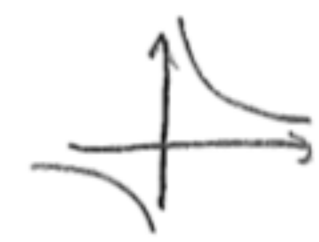
Theorem: If f is differentiable at $x = c$, then f is also continuous at $x = c$.
Equivalently, if f is not continuous at $x = c$, then f is not differentiable at $x = c$.

Note: If f is not differentiable at $x = c$, then f may or may not be continuous at $x = c$.

(a) $f(x) = |x|$ is not differentiable at $x = 0$, but it is continuous at $x = 0$. "sharp point" →



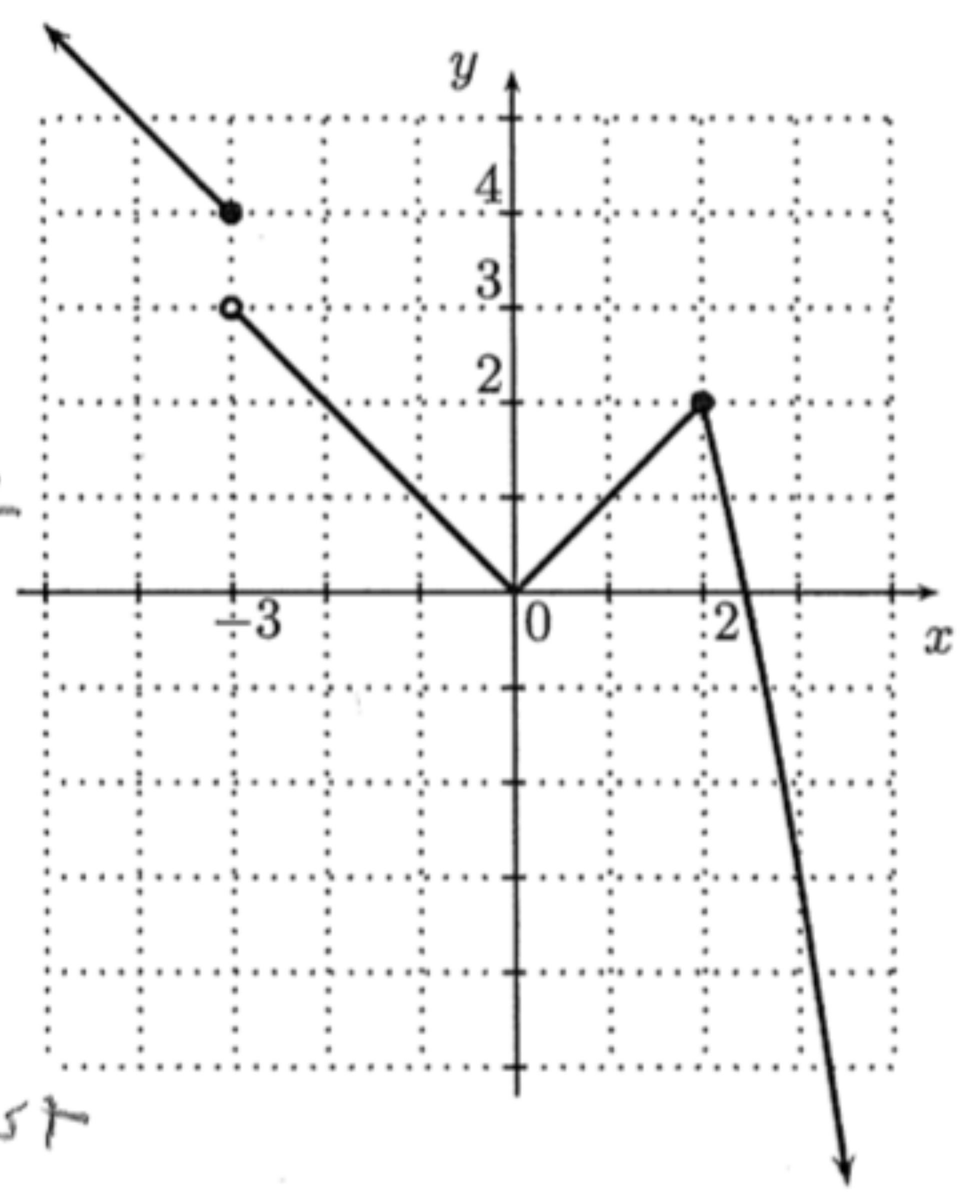
(b) $f(x) = \dfrac{1}{x}$ is not differentiable at $x = 0$ and it is not continuous at $x = 0$.



Look at the graph of the Dow Jones Industrial Average for a real life example of a function that is continuous everywhere but which has many points of non-differentiability.

Example 5: Determine the x values where the derivative of the function is not defined (that is the points where the function is not differentiable). Is the function continuous at those points?

$$g(x) = \begin{cases} -x+1 & \text{if } x \le -3 \\ |x| & \text{if } -3 < x < 2 \\ -x^2+6 & \text{if } x \ge 2 \end{cases}$$



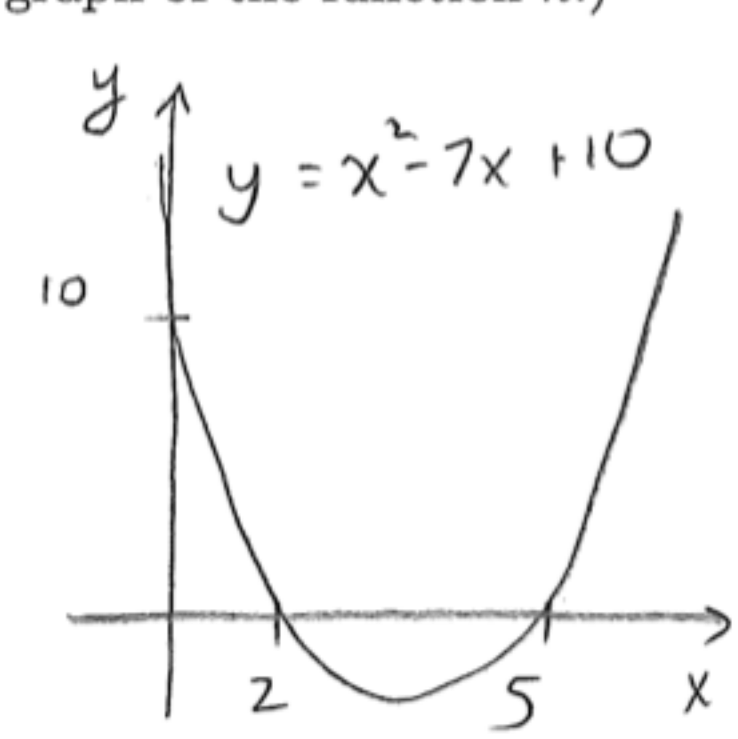
$x = -3$: not continuous, not differentiable

$x = 0$, $x = 2$: is continuous, but not differentiable.

$f'(-3)$, $f'(0)$, $f'(2)$ all do not exist

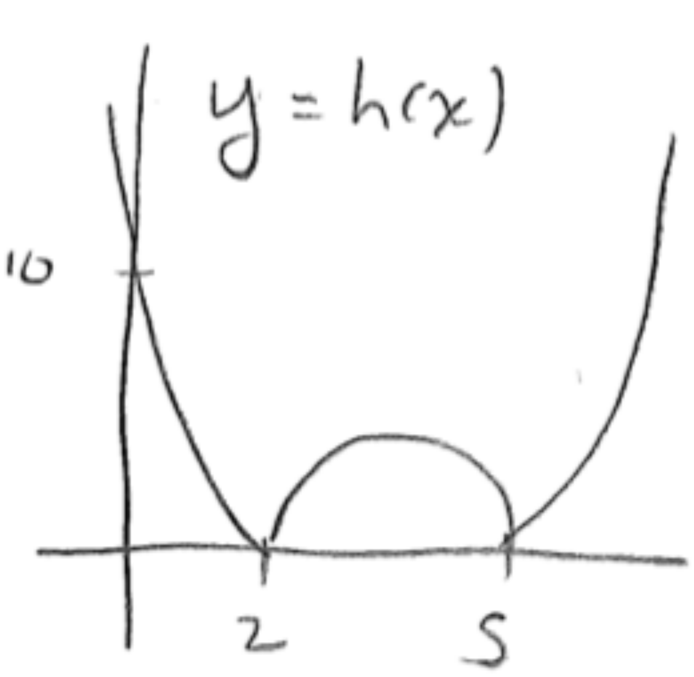
Example 6: Determine the x values where the derivative of $h(x) = |x^2 - 7x + 10|$ is not defined. Is $h(x)$ continuous at those points? (**Hint:** first draw the graph of the equation $y = x^2 - 7x + 10$ and then draw the graph of the function h.)

$= (x-5)(x-2)$



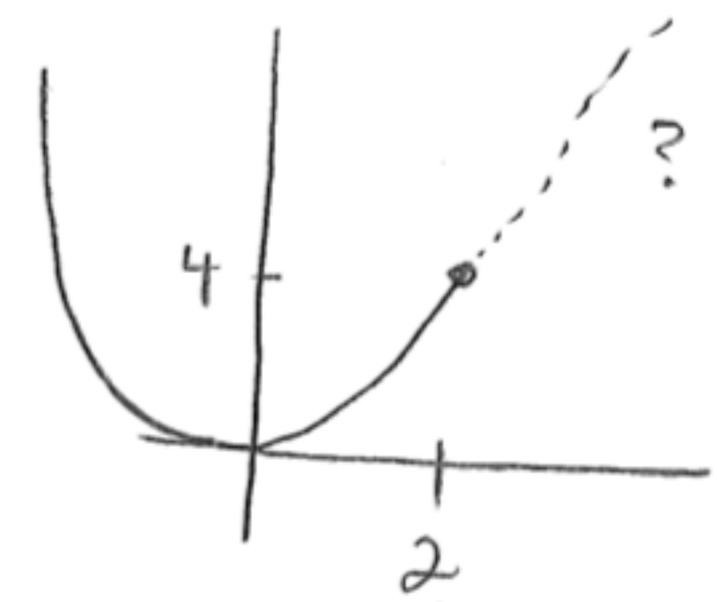
→

absolute value turns negative values positive



$h(x)$ is continuous everywhere, but is not differentiable at $x = 2$, $x = 5$.

Example 7: Let



$$f(x) = \begin{cases} x^2, & \text{if } x \le 2; \\ mx + b, & \text{if } x > 2. \end{cases}$$

Find the values of m and b that make f differentiable at $x = 2$.

We need a line so that both the slope (derivative) at $x=2$ and the y-value at $x=2$ agree with $y=x^2$.

Continuity: $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 = 2^2 = 4.$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (mx+b) = 2m+b$

thus, $2m + b = 4$

Slope: From the left, for $x < 2$, $f'(x) = 2x$

at $x = 2$, $f'(2) = 2(2) = 4$.

From the right, for $x > 2$, $f'(x) = m$.

Thus: $\boxed{m = 4}$

$2m + b = 4$

$\Rightarrow 2(4) + b = 4$

$8 + b = 4$

$\boxed{b = -4}$

► **Using the limit definition to compute derivatives:**

The instantaneous rate of change of a function f with respect to x at a general point x is called the *derivative of f at x* and is denoted with $f'(x)$:

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.}$$

Next, we use this definition of the derivative to learn how to differentiate functions of the following types:

$$f(x) = (x+\alpha)^2 \qquad f(x) = \frac{1}{x+\alpha} \qquad f(x) = \sqrt{x+\alpha} \qquad f(x) = (x+\alpha)^3$$

where α is an arbitrary real number. For each type of function, the calculation of the limit has to be treated with a different technique.

Special product formulas: The powers of certain binomials occur so frequently that we should memorize the following formulas. We can verify them by performing the multiplications.

If A and B are any real numbers or algebraic expressions, then:

(1.) $(A+B)^2 = A^2 + 2AB + B^2$

(2.) $(A-B)^2 = A^2 - 2AB + B^2$

(3.) $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$

(4.) $(A-B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$

Example 8: Let $f(x) = (x+4)^2$.

(a) Find and simplify the **difference quotient** $\dfrac{f(x+h) - f(x)}{h}$.

(b) Use your answer from part (a) to find $f'(x)$.

(c) Find $f'(5)$. Write the equation of the tangent line to the graph of f at $x = 5$ in the form $y = mx + b$.

(a) $f(x) = (x+4)^2 = (x+4)(x+4) = x^2 + 8x + 16.$

$* \; f(x+h) = (x+h+4)^2 = (x+h+4)(x+h+4)$ ← distribute:

$= x(x+h+4) + h(x+h+4) + 4(x+h+4)$

$= x^2 + xh + 4x + xh + h^2 + 4h + 4x + 4h + 16$ combine like terms.

$= x^2 + 2xh + 8x + h^2 + 8h + 16$

$$\frac{f(x+h) - f(x)}{h} = \frac{x^2 + 2xh + 8x + h^2 + 8h + 16 - (x^2 + 8x + 16)}{h}$$

$$= \frac{2xh + h^2 + 8h}{h} = \frac{\cancel{h}(2x + h + 8)}{\cancel{h}} = \boxed{2x + h + 8}$$

(b) $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (2x + h + 8) = 2x + 0 + 8$$

$$= \boxed{2x + 8}$$

(c) $f'(5) = 2(5) + 8 = 10 + 8 = 18$ ← slope

y-value at $x = 5$ is $f(5) = (5+4)^2 = 9^2 = 81$

$y - y_1 = m(x - x_1) \Rightarrow y - 81 = 18(x-5)$

$y - 81 = 18x - 90$

$$\boxed{y = 18x - 9}$$

$*$ Alternate method: begin by writing $f(x) = (x+4)^2 = (x+4)(x+4)$

$= x^2 + 8x + 16$,

then evaluate $f(x+h)$

Example 9: Let $f(x) = \dfrac{1}{x+3}$.

(a) Find and simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$.

(b) Use your answer from part (a) to find $f'(x)$.

(c) Find $f'(5)$.

(a) $f(x+h) = \dfrac{1}{x+h+3}$

$$\frac{f(x+h)-f(x)}{h} = \frac{\dfrac{1}{x+h+3} - \dfrac{1}{x+3}}{h}$$

Find a common denominator

$$= \frac{\dfrac{1}{x+h+3}\cdot\dfrac{(x+3)}{(x+3)} - \dfrac{1}{x+3}\cdot\dfrac{(x+h+3)}{(x+h+3)}}{h}$$

$$= \frac{\dfrac{x+3-(x+h+3)}{(x+h+3)(x+3)}}{\dfrac{h}{1}} = \frac{-h}{(x+h+3)(x+3)}\cdot\frac{1}{h}$$

(dividing by $\frac{h}{1}$ is like multiplying by $\frac{1}{h}$)

$$= \boxed{\frac{-1}{(x+h+3)(x+3)}}$$

(b) $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{-1}{(x+h+3)(x+3)} = \frac{-1}{(x+0+3)(x+3)}$$

$$= \frac{-1}{(x+3)^2}$$

(c) $$f'(5) = \frac{-1}{(5+3)^2} = \boxed{\frac{-1}{64}}$$

Example 10: Let $f(x) = \sqrt{x-2}$.

(a) Find and simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$.

(b) Use your answer from part (a) to find $f'(x)$.

(c) Find $f'(6)$ and $f'(11)$.

(a) $f(x+h) = \sqrt{x+h-2}$

multiply and divide by the conjugate

$$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x+h-2} - \sqrt{x-2}}{h} \cdot \frac{\sqrt{x+h-2} + \sqrt{x-2}}{\sqrt{x+h-2} + \sqrt{x-2}}$$

$$= \frac{(\sqrt{x+h-2})^2 - (\sqrt{x-2})^2}{h(\sqrt{x+h-2} + \sqrt{x-2})}$$

using $(a-b)(a+b) = a^2 - b^2$

$$= \frac{x+h-2 - (x-2)}{h(\sqrt{x+h-2} + \sqrt{x-2})}$$

since $(\sqrt{x})^2 = x$

$$= \frac{h}{h(\sqrt{x+h-2} + \sqrt{x-2})} = \boxed{\frac{1}{\sqrt{x+h-2} + \sqrt{x-2}}}$$

(b) $$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{1}{\sqrt{x+h-2} + \sqrt{x-2}}$$

$$= \frac{1}{\sqrt{x+0-2} + \sqrt{x-2}} = \boxed{\frac{1}{2\sqrt{x-2}}}$$

(c) $$f'(6) = \frac{1}{2\sqrt{6-2}} = \frac{1}{2\sqrt{4}} = \boxed{\frac{1}{4}}$$

$$f'(11) = \frac{1}{2\sqrt{11-2}} = \frac{1}{2\sqrt{9}} = \boxed{\frac{1}{6}}$$

Example 11: Let $g(x) = x^3$.

(a) Find and simplify the difference quotient $\dfrac{g(x+h) - g(x)}{h}$.

(b) Use your answer from part (a) to find $g'(x)$.

(a) $g(x+h) = (x+h)^3 = (x+h)(x^2+2xh+h^2)$

distribute: $= \underbrace{x^3 + 2x^2h + xh^2} + \underbrace{xh^2 + 2xh^2 + h^3}$

combine terms: $= x^3 + 3x^2h + 3xh^2 + h^3$

$$\frac{g(x+h) - g(x)}{h} = \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \frac{h(3x^2 + 3xh + h^2)}{h}$$

$$= \boxed{3x^2 + 3xh + h^2}$$

(b) $$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2)$$

$$= 3x^2 + 3x(0) + 0^2$$

$$\boxed{= 3x^2}$$

Example 12: If $f(x) = \dfrac{-2}{x-3}$, then $\dfrac{f(x+h)-f(x)}{h} = \dfrac{A}{(x-3)(x+Bh+C)}$. Find A, B, and C.

$$\frac{f(x+h)-f(x)}{h} = \frac{\dfrac{-2}{x+h-3} - \left(\dfrac{-2}{x-3}\right)}{h}$$

careful with negatives

$$= \frac{\dfrac{-2}{x+h-3}\cdot\dfrac{x-3}{x-3} + \dfrac{2}{x-3}\cdot\dfrac{x+h-3}{x+h-3}}{\dfrac{h}{1}}$$

$$= \frac{-2(x-3) + 2(x+h-3)}{(x-3)(x+h-3)} \cdot \frac{1}{h}$$

$$= \frac{-2x+6+2x+2h-6}{(x-3)(x+h-3)} \cdot \frac{1}{h}$$

$$= \frac{2h}{(x-3)(x+h-3)} \cdot \frac{1}{h} = \frac{2}{(x-3)(x+h-3)}$$

Matching with the expression above, we have $\boxed{A = 2, \quad B = 1, \quad C = -3}$

Example 13: Suppose that $\dfrac{f(x+h)-f(x)}{h} = \dfrac{-2h(x+2)-h^2}{h(x+h+2)^2(x+2)^2}$.
Find the slope m of the tangent line at $x=1$.

$m = f'(1)$. We know $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \dfrac{-2h(x+2)-h^2}{h(x+h+2)^2(x+2)^2}$

test $h=0$, get $\frac{0}{0}$:
do more work!
Factor out h from numerator,
then cancel

$= \lim_{h\to 0} \dfrac{\cancel{h}\left(-2(x+2)-h\right)}{\cancel{h}(x+h+2)^2(x+2)^2}$

$= \lim_{h\to 0} \dfrac{-2(x+2)-h}{(x+h+2)^2(x+2)^2} = \dfrac{-2(x+2)-0}{(x+0+2)^2(x+2)^2}$

Now let $x=1$ to find $m = f'(1)$: $\dfrac{-2(1+2)}{(1+2)^2(1+2)^2}$

$= \dfrac{-2(3)}{3^2\cdot 3^2} = \dfrac{-6}{81} = \boxed{\dfrac{-2}{27}}$

Example 14: Suppose that

$$\frac{f(x+h)-f(x)}{h} = 3x+2h-1 \quad \text{and} \quad f(1)=4.$$ ← y-value is 4.

Find the equation of the tangent line to the graph of $y=f(x)$ at $x=1$.

$f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(3x+2h-1) = 3x+2(0)-1$
$= 3x-1$.

$m = f'(1) = 3(1)-1 = 2$.

Equation: $y-y_1 = m(x-x_1) \Rightarrow y-4 = 2(x-1)$

$y-4 = 2x-2$
$+4 \qquad +4$

$\boxed{y = 2x+2}$

▶ **Approximating a derivative:** Considering how much effort was required to compute the derivative of a seemingly simple function like $g(x) = \sqrt{x-2}$, you may think that it will be almost impossible to compute the derivative of functions like $G(x) = e^{-x^2}$ or $h(x) = (x^2+17)^9$. In the next two chapters we will learn algebraic formulas which will help compute derivatives of lots of functions. We end this chapter by suggesting an alternative method for finding derivatives of more complex functions: numeric approximation.

Example 14: Let $f(x) = (x+7)^5$. Approximate $f'(2)$. ≈ 32805

h	-0.1	-0.01	-0.001	-0.0001	-0.00001	0.00001	0.0001	0.001	0.01	0.1
$\dfrac{f(2+h)-f(2)}{h} = \dfrac{(h+9)^5-9^5}{h}$	32084.05	32732.18	32797.71	32804.27	32804.92	32805.07	32805.72	32812.29	32877.98	33542.14

Example 15: Let $g(x) = \ln(x)$. Approximate $g'(2)$. $\approx 0.5 = \frac{1}{2}$

h	-0.1	-0.01	-0.001	-0.0001	-0.00001	0.00001	0.0001	0.001	0.01	0.1
$\dfrac{g(2+h)-g(2)}{h} = \dfrac{\ln(2+h)-\ln(2)}{h}$	0.5129	0.5012	0.5001	0.5000	0.5000	0.4999	0.4999	0.4998	0.4987	0.4879

Example 16: Let $f(x) = 3^x$. Approximate $f'(1)$. ≈ 3.2958

h	-0.1	-0.01	-0.001	-0.0001	-0.00001	0.00001	0.0001	0.001	0.01	0.1
$\dfrac{f(1+h)-f(1)}{h} = \dfrac{3\cdot(3^h-1)}{h}$	3.1212	3.2777	3.2940	3.2956	3.2958	3.2958	3.2960	3.2976	3.3140	3.4836

Example 17: Let $g(x) = |5x|$. Approximate $g'(0)$. = DNE because LHL $\neq$ RHL

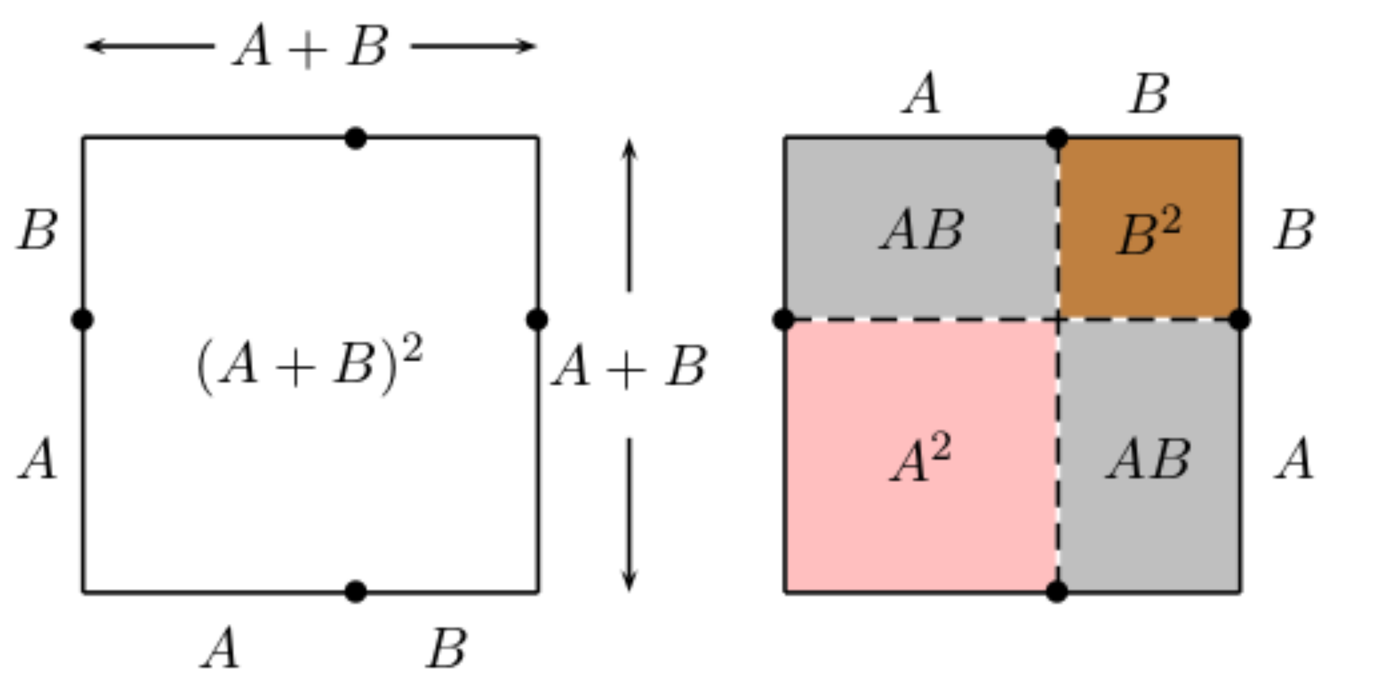
h	-0.1	-0.01	-0.001	-0.0001	-0.00001	0.00001	0.0001	0.001	0.01	0.1
$\dfrac{g(0+h)-g(0)}{h} = \dfrac{|5h|}{h}$	-5	-5	-5	-5	-5	5	5	5	5	5

The special product formulas on page 30 follow a predictable pattern that deserves further inspection.

Visualizing a formula:

Many of these product formulas can be seen as geometrical facts about length, area, and volume. The ancient Greeks always interpreted algebraic formulas in terms of geometric figures.

For example, the figure below



$$(A+B)^2 = A^2 + 2AB + B^2$$

shows how the formula for the square of a binomial (formula **1**) can be interpreted as a fact about areas of squares and rectangles.

Pascal's triangle: The coefficients (without sign) of the expansion of a binomial of the form $(a \pm b)^n$ can be read off the n-th row of the following 'triangle' named **Pascal's triangle** (after Blaise Pascal, a 17th century French mathematician and philosopher).

To build the triangle, start with '1' at the top, then continue placing numbers below it in a triangular way. Each number is simply obtained by adding the two numbers directly above it.