SOLUTIONS

MA123, Chapter 6: Exponential and logarithmic functions

Chapter Goals:	• Review properties of exponential and logarithmic functions. • Learn how to differentiate exponential and logarithmic functions. • Learn about exponential growth and decay phenomena.
Assignments:	Assignment 10 Assignment 11 Assignment 12

Recall the following from Chapter 1.

Exponential notation: If a is any real number and n is a positive integer, then the **n-th power** of a is

$$a^n = \underbrace{a \cdot a \cdot \ \dots \ \cdot a}_{n \text{ times}}$$

The number a is called the **base** whereas n is called the **exponent**.

The first and second laws of exponents below allow us to define a^n for any integer n.

Now, we want to define, for instance, $a^{1/3}$ in a way that is consistent with the laws of exponents. We would like:

$$\left(a^{1/3}\right)^3 = a^{(1/3)3} = a^1 = a; \qquad \text{thus} \qquad a^{1/3} = \sqrt[3]{a}$$

So, by the definition of nth root, we have:

$$\boxed{a^{1/n} = \sqrt[n]{a}}$$

Definition of rational exponents: For any rational exponent m/n in lowest terms, where m and n are integers and $n > 0$, we define

$$a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m \qquad \text{or equivalently}$$

$$a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}$$

If n is even we require that $a \geq 0$.

In the table below, a and b are real numbers ($\neq 0$ if needed) and the exponents x and y are rational numbers.

Laws of exponents:

(1.) $a^0 = 1$	(3.) $a^x a^y = a^{x+y}$	(6.) $(ab)^x = a^x b^x$
(2.) $a^{-x} = \dfrac{1}{a^x}$	(4.) $\dfrac{a^x}{a^y} = a^{x-y}$	(7.) $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$
	(5.) $(a^x)^y = a^{xy}$	

Now, let $a > 0$ be a positive number with $a \neq 1$. Thus far a^x is defined for x a rational number. So, what does, for instance, $5^{\sqrt{2}}$ mean? When x is irrational, we successively approximate x by rational numbers. For instance, as

$$\sqrt{2} \approx 1.41421\ldots$$

we successively approximate $5^{\sqrt{2}}$ with

$$5^{1.4}, \quad 5^{1.41}, \quad 5^{1.414}, \quad 5^{1.4142}, \quad 5^{1.41421}, \ldots$$

In practice, we simply use our calculator and find out

$$5^{\sqrt{2}} \approx 9.73851\ldots$$

▶ **Exponential functions:**

Let $a > 0$ be a positive number with $a \neq 1$. The **exponential function with base** a is defined by

$$f(x) = a^x$$

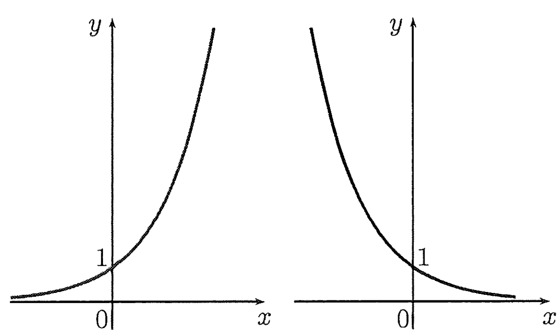
for all real numbers x.

Graphs of exponential functions:

The **exponential function**

$$f(x) = a^x \qquad (a > 0, \ a \neq 1)$$

has domain $\mathbb{R}$ and range $(0, \infty)$. The graph of $f(x)$ has one of these shapes:



$f(x) = a^x$ for $a > 1$ $\qquad$ $f(x) = a^x$ for $0 < a < 1$

The most important base is the number denoted by the letter e. The number e is defined as

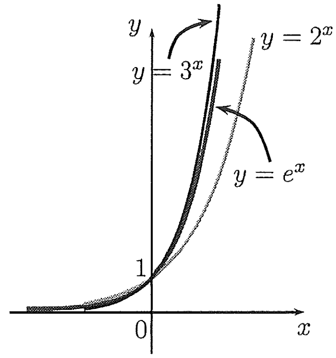
$$\boxed{e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}$$

Correct to five decimal places (note that e is an irrational number), $e \approx 2.71828$.

The natural exponential function:

The **natural exponential function** is the exponential function
$$f(x) = e^x$$
with base e. It is often referred to as *the* exponential function.

Since $2 < e < 3$, the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$.



n	$\left(1+\frac{1}{n}\right)^n$
1	2.00000
5	2.48832
10	2.59374
100	2.70481
1,000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

▶ **Inverse of a function:** Recall that two functions $f(x)$ and $g(x)$ are said to be inverses of each other if

$$f(g(x)) = x \qquad \text{and} \qquad g(f(x)) = x.$$

Intuitively, inverse function pairs are in some sense "opposites." The three most familiar inverse function pairs are: "addition and subtraction," "multiplication and division," and "square and square root":

- $g(t) = t - c$ and $h(t) = t + c$ are inverse to each other, for any real number c.
- $g(t) = t/k$ and $h(t) = k \cdot t$ are inverse to each other, for any real number $k \neq 0$.
- $g(t) = \sqrt{t}$ and $h(t) = t^2$ are inverse to each other, provided $t \geq 0$.

▶ **Logarithmic functions:** Every exponential function $f(x) = a^x$, with $a > 0$ and $a \neq 1$, is a one-to-one function by the *horizontal line test.* Thus, it has an inverse function. The inverse function $f^{-1}(x)$ is called the *logarithmic function with base a* and is denoted by $\log_a x$.

Definition: Let a be a positive number with $a \neq 1$. The **logarithmic function** with base a, denoted by $\log_a$, is defined by

$$y = \log_a(x) \iff a^y = x.$$

In other words, $\log_a(x)$ is the exponent to which the base a must be raised to give x.

Properties of logarithms:

(1.) $\log_a(1) = 0$ (3.) $\log_a(a^x) = x$

(2.) $\log_a(a) = 1$ (4.) $a^{\log_a(x)} = x$

Since logarithms are 'exponents', the laws of exponents give rise to the laws of logarithms:

Let a be a positive number, with $a \neq 1$. Let A, B and C be any real numbers with $A > 0$ and $B > 0$.

Laws of logarithms:

(1.) $\log_a(AB) = \log_a(A) + \log_a(B)$;

(2.) $\log_a\left(\frac{A}{B}\right) = \log_a(A) - \log_a(B)$;

(3.) $\log_a(A^C) = C\,\log_a(A)$.

Change of base:

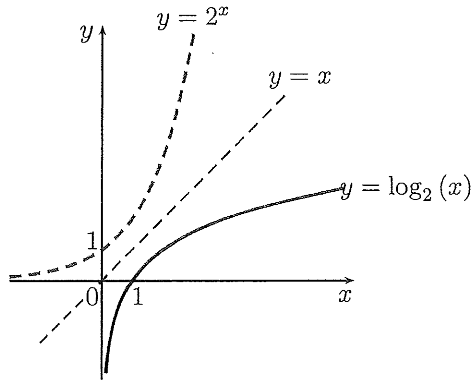
For some purposes, we find it useful to change from logarithms in one base to logarithms in another base. One can prove that:

$$\boxed{\log_b x = \frac{\log_a(x)}{\log_a(b)}}$$

Remark: If a one-to-one function f has domain A and range B, then its inverse function f^{-1} has domain B and range A. THUS, the function $y = \log_a(x)$ is defined for $x > 0$ and has range equal to $\mathbb{R}$. More precisely:

Graphs of logarithmic functions:

The graph of $f^{-1}(x) = \log_a(x)$ is obtained by reflecting the graph of $f(x) = a^x$ in the line $y = x$.
(The picture below shows a typical case with $a > 1$.)



The point $(1, 0)$ is on the graph of $y = \log_a(x)$ (as $\log_a(1) = 0$) and the y-axis is a vertical asymptote.

Common logarithms:

The logarithm with base 10 is called the **common logarithm** and is denoted by omitting the base:

$$\boxed{\log(x) := \log_{10}(x).}$$

Natural logarithms: Of all possible bases a for logarithms, it turns out that the most convenient choice for the purposes of Calculus is the number e.

Definition: The logarithm with base e is called the **natural logarithm** and is denoted by ln:

$$\boxed{\ln(x) := \log_e(x).}$$

We recall again that, by the definition of inverse functions, we have

$$y = \ln(x) \qquad \Longleftrightarrow \qquad e^y = x.$$

Properties of natural logarithms:

(1.) $\ln(1) = 0$ (3.) $\ln(e^x) = x$

(2.) $\ln(e) = 1$ (4.) $e^{\ln(x)} = x$

for h values very close to zero, we find values of $\frac{e^h - 1}{h}$ very close to 1

▶ **Derivatives**

Fact: By filling the table below we can convince ourselves that $\boxed{\lim_{h \to 0} \frac{e^h - 1}{h} = 1.}$

h	-0.1	-0.01	-0.001	-0.0001	-0.00001	0.00001	0.0001	0.001	0.01	0.1
$\frac{e^h - 1}{h}$	.9516	.995	.9995	.99995	.999995	1.000005	1.00005	1.0005	1.005	1.0517

Now, let $f(x) = e^x$. Using the definition of the derivative and the rules for exponential functions, we have that

$$\frac{d}{dx}(e^x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x e^h - e^x}{h} = e^x \left(\lim_{h \to 0} \frac{e^h - 1}{h} \right) = e^x$$

Theorem:

$$\boxed{\frac{d}{dx}(e^x) = e^x \qquad \text{or} \qquad (e^x)' = e^x.}$$

Moreover, it follows by applying the chain rule that

$$\boxed{\frac{d}{dx}(e^{g(x)}) = e^{g(x)} \frac{d}{dx}(g(x)) \qquad \text{or} \qquad (e^{g(x)})' = e^{g(x)}\, g'(x).}$$

We can use the derivative of e^x and the relationship between the exponential and the natural logarithmic functions to find the derivative of the function $\ln(x)$. Namely, take the derivative with respect to x of both sides of $e^{\ln(x)} = x$. We obtain

$$\frac{d}{dx}(e^{\ln x}) = \frac{d}{dx}(x) \qquad \text{or} \qquad e^{\ln x}\frac{d}{dx}(\ln x) = 1 \qquad \text{or} \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Theorem:

$$\boxed{\frac{d}{dx}(\ln(x)) = \frac{1}{x} \qquad \text{or} \qquad (\ln(x))' = \frac{1}{x}.}$$

Moreover, it follows by applying the chain rule that

$$\boxed{\frac{d}{dx}(\ln(g(x))) = \frac{1}{g(x)}\frac{d}{dx}(g(x)) \qquad \text{or} \qquad (\ln(g(x)))' = \frac{g'(x)}{g(x)}.}$$

What about more general derivatives? Observe that we have the identities

$$a^x = e^{\ln(a^x)} = e^{x\ln(a)} \qquad \log_a(x) = \frac{\ln(x)}{\ln(a)}.$$

Thus using the previous results we obtain the following formulas for the derivatives of general exponential and logarithmic functions

$$\boxed{\frac{d}{dx}(a^x) = a^x\ln(a) \qquad \text{and} \qquad \frac{d}{dx}(\log_a(x)) = \frac{1}{x\ln(a)}.}$$

Note: Let us consider the function $f(x) = 3^x$. In Example 16 of Chapter 4, we saw that an approximation for $f'(1)$ was given by the value 3.2958. Using the above formula we have that $f'(x) = 3^x\ln(3)$, so that the exact value for $f'(1)$ is $3\ln(3) = \ln(27)$.

Example 1: Find the derivative with respect to x of $f(x) = e^{4x}$. Evaluate $f'(x)$ at $x = 1/4$. Compute $f''(x)$, $f'''(x)$ and $f^{(10)}(x)$. Can you predict what the nth derivative $f^{(n)}(x)$ of $f(x)$ looks like?

$f'(x) = 4e^{4x}$ $\quad f'(\frac{1}{4}) = 4e^{4(\frac{1}{4})} = 4e^1 = 4e$

$f''(x) = 4^2e^{4x}$, $\quad f'''(x) = 4^3e^{4x}$, $\ldots$, $\quad f^{(10)}(x) = 4^{10}e^{4x}$,

$\boxed{f^{(n)}(x) = 4^n e^{4x}}$ each time we differentiate, we multiply by another factor of 4.

Example 2: Find the derivative with respect to x of $g(x) = x^2e^x$. Evaluate $g'(x)$ at $x = 1$.

$g'(x) = x^2 \cdot e^x + e^x(2x)$ ← product rule

$g'(1) = 1^2 \cdot e^1 + e^1(2 \cdot 1) = e + 2e = \boxed{3e}$

Example 3: Suppose $f(t) = e^{\sqrt{3t-4}}$. Find $\frac{df}{dt}$.

$$f(t) = e^{(3t-4)^{1/2}}$$

$$\frac{df}{dt} = f'(t) = \underbrace{e^{(3t-4)^{\frac{1}{2}}}}_{\text{e rule}} \cdot \underbrace{\tfrac{1}{2}(3t-4)^{-\frac{1}{2}}}_{\text{power rule}} \cdot \underbrace{3}_{\text{inside}}$$

Example 4: Suppose $f(3) = 4$, $f'(3) = -6$, $g(3) = 5$, $g'(3) = 2$, and $H(x) = f(x)e^{g(x)}$. ← needs product rule

Find the derivative $\left.\frac{dH}{dx}\right|_{x=3}$.

copy · deriv + copy · deriv

$$\frac{dH}{dx} = H'(x) = f(x) \cdot e^{g(x)} \cdot g'(x) + e^{g(x)} \cdot f'(x)$$

$$\left.\frac{dH}{dx}\right|_{x=3} = H'(3) = f(3) \cdot e^{g(3)} g'(3) + e^{g(3)} f'(3)$$

$$= 4 \cdot e^5 \cdot 2 + e^5 \cdot (-6) = 8e^5 - 6e^5 = \boxed{2e^5}$$

Example 5: Find the derivative with respect to x of $f(x) = x\ln(x)$. ← needs product rule

copy deriv + copy deriv

$$f'(x) = \boxed{x \cdot \frac{1}{x} + \ln(x) \cdot 1}$$ ← not simplified

$$= \boxed{1 + \ln x}$$ simplified

Example 6: Find the derivative with respect to x of $y = \ln(5x+1)$. ← does NOT need product rule

$$y' = \boxed{\frac{1}{5x+1} \cdot 5}$$ ← not simplified

$$= \boxed{\frac{5}{5x+1}}$$ ← simplified

outside inside

Example 7: Find $\frac{d}{dx}\left(\ln\left(3x^4 - 7x^2 + 5\right)\right)$.

$$= \boxed{\frac{1}{3x^4 - 7x^2 + 5} \cdot (12x^3 - 14x)}$$

$$= \boxed{\frac{12x^3 - 14x}{3x^4 - 7x^2 + 5}}$$

1 2 3

Example 8: Find the derivative with respect to x of $f(x) = \ln\left(\ln\left(\ln\left(x\right)\right)\right)$.

$$f'(x) = \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{1}{x}$$

outside ln | middle | innermost ln x

Example 9: Find the derivative with respect to x of $h(x) = e^{x^2 + 3\ln(x)}$.

$$h'(x) = \boxed{\underbrace{e^{x^2 + 3\ln x}}_{\text{e rule}} \cdot \underbrace{\left(2x + 3 \cdot \frac{1}{x}\right)}_{\text{inside}}}$$

$$= e^{x^2 + 3\ln x}\left(2x + \frac{3}{x}\right)$$

Example 10: Suppose $f(8) = 11$, $f'(8) = 5$, $g(2) = 1$, $g'(2) = 3$, and $H(x) = f\left(x^3 + \ln\left(g(x)\right)\right)$.

Find the derivative $\left.\frac{dH}{dx}\right|_{x=2}$.

outside inside

$$H'(x) = f'\left(x^3 + \ln(g(x))\right) \cdot \left(3x^2 + \frac{1}{g(x)} \cdot g'(x)\right)$$

$$H'(2) = f'\left(2^3 + \ln(g(2))\right) \cdot \left(3 \cdot 2^2 + \frac{1}{g(2)} \cdot g'(2)\right)$$

$$= f'(8 + \underbrace{\ln(1)}_{0}) \cdot \left(12 + \frac{3}{1}\right)$$

$$= 5 \cdot (15) = \boxed{75}$$

$\boxed{\text{Exponential growth and decay}}$

Let $Q(t)$ denote the amount of a quantity as a function of time. We say that $Q(t)$ grows exponentially as a function of time if

$$Q(t) = Q_0\, e^{rt},$$

where Q_0 and r are positive constants that depend on the specific problem and t denotes time. When $t = 0$, we see that

$$Q(0) = Q_0\, e^{r \cdot 0} = Q_0 \cdot 1 = Q_0.$$

Thus Q_0 denotes the amount of the quantity at $t = 0$. In other words, Q_0 is the initial amount of the quantity at time $t = 0$. Note that $Q(t) > 0$ because $Q_0 > 0$ and $e^{rt} > 0$.

Taking the derivative and using the chain rule, we see that

$$Q'(t) = Q_0 \cdot r \cdot e^{rt} = r\,(Q_0\, e^{rt}) = r\, Q(t).$$

Since $\boxed{Q'(t) = r\, Q(t),}$ it follows that if a quantity grows exponentially, then its rate of growth is proportional to the quantity present, and the proportionality constant is given by r. Since $r > 0$ and $Q(t) > 0$, we have $Q'(t) > 0$, as expected because $Q(t)$ is increasing.

Some quantities decrease exponentially. In this case we have $Q(t) = Q_0\, e^{-rt}$, where Q_0 and r are positive constants. Note that we have $Q(0) = Q_0$ and

$$Q'(t) = Q_0 \cdot (-r) \cdot e^{-rt} = -r\,(Q_0\, e^{-rt}) = -r\, Q(t).$$

Thus $\boxed{Q'(t) = -r\, Q(t).}$ We see that $Q'(t) < 0$ because $-r < 0$ and $Q(t) > 0$. Thus the rate of increase of $Q(t)$ is proportional to the quantity present, and the proportionalityconstant is given by $-r$.

Suppose that a function $g(x)$ satisfies the property that the slope of the tangent line to the graph of $y = g(x)$ at any point P is proportional to the y-coordinate of P, i.e., $g'(x_P) = r \cdot g(x_P)$. Then it can be shown that there are constants C and r such that $g(x) = Ce^{rx}$. In fact, r is the constant of proportionality because $g'(x) = rCe^{rx} = rg(x)$.

y'

Example 11: The graph of a function $g(x)$ passes through the point $(0, 5)$. Suppose that the slope of the tangent line to the graph of $y = g(x)$ at any point P is 7 times the y-coordinate of P. Find $g(2)$.

We are given that $y' = 7y$, so we know the function is of the form $g(x) = g_0\, e^{7x}$

Since the graph goes through $(0,5)$, we know $g(0) = 5$, so $5 = g_0\, e^0 \Rightarrow g_0 = 5$. The function is $g(x) = 5e^{7x}$.

Thus, $g(2) = 5e^{7(2)} = \boxed{5e^{14}}$

Applications

Many processes that occur in nature, such as calculation of bank interest, population growth, radioactive decay, heat diffusion, and numerous others, can be modeled using exponential functions. Logarithmic functions are used in models for the loudness of sounds, the intensity of earthquakes, and many other phenomena.

Compound interest is calculated by the formula:

$$\boxed{P(t) = P_0\left(1+\frac{r}{n}\right)^{nt}}$$

where

$P(t)$	$=$	principal after t years
P_0	$=$	initial principal
r	$=$	interest rate per year
n	$=$	number of times interest is compounded per year
t	$=$	number of years

Continuously compounded interest

is calculated by the formula:

$$\boxed{P(t) = P_0\, e^{rt}}$$

where

$P(t)$	$=$	principal after t years
P_0	$=$	initial principal
r	$=$	interest rate per year
t	$=$	number of years

Proof: The interest paid increases as the number n of compounding periods increases. If $m = \frac{n}{r}$, then:

$$P_0\left(1+\frac{r}{n}\right)^{nt} = P_0\left[\left(1+\frac{r}{n}\right)^{n/r}\right]^{rt} = P_0\left[\left(1+\frac{1}{n/r}\right)^{n/r}\right]^{rt} = P_0\left[\left(1+\frac{1}{m}\right)^{m}\right]^{rt}.$$

As n becomes large, m also becomes large. Since $\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = e$ we obtain the formula for continuously compounded interest.

Example 12: If \$10,000 is invested at an interest rate of 6%, find the value of the investment at the end of 8 years if the interest is compounded continuously. $r = .06$

$$P(8) = 10000\,e^{.06(8)}$$

$$\boxed{= \$10000\,e^{.48}} \approx \boxed{\$16160.74}$$

Example 13: How many years will it take an investment to quadruple in value if the interest is compounded continuously at a rate of 7%? $\leftarrow r = .07$

Suppose we invest \$1. How long until we have \$4?

Solve for t: $4 = 1\,e^{.07t}$

$$\ln 4 = \ln e^{.07t}$$

$$\ln 4 = .07t$$

$$\boxed{t = \frac{\ln 4}{.07} \text{ years}} \approx \boxed{19.8 \text{ years}}$$

$r = .05$

Example 14: An amount of P_0 dollars is invested at 5% interest compounded continuously. Find P_0 if at the end of 10 years the value of the investment is \$20,000.

$$20000 = P_0 e^{.05(10)}.$$

divide both sides by $e^{.5}$

$$P_0 = \boxed{\frac{\$20000}{e^{.5}}}$$

$$\boxed{\approx \$12130.61}$$

Exponential models of population growth:

The formula for population growth of several species is the same as that for continuously compounded interest. In fact in both cases the rate of growth of a population (or an investment) per time period is proportional to the size of the population (or the amount of an investment).

Remark: Biologists sometimes express the growth rate r in terms of the **doubling-time** t_0, the time required for the population to double in size: $r = \dfrac{\ln(2)}{t_0}$.

Exponential growth model If P_0 is the initial size of a population that experiences **exponential growth**, then the population $P(t)$ at time t increases according to the model

$$\boxed{P(t) = P_0 e^{rt}}$$

where r is the relative rate of growth of the population (expressed as a proportion of the population).

Note: If t_0 denotes the doubling-time of a population, we can rewrite the expression for $P(t)$ as follows

$$P(t) = P_0\, e^{rt} = P_0\, e^{(\ln(2)/t_0)\cdot t} = P_0 \left(e^{\ln(2)}\right)^{t/t_0} = P_0\, 2^{t/t_0}.$$

Example 15: A bacteria culture starts with 2,000 bacteria and the population triples after 5 hours. If we express the number of bacteria after t hours as

$$y(t) = a\, e^{bt}$$

"a" will be the initial value, 2000.

find a and b. To solve for b:

After 5 hours, there are 6000 cells: $6000 = 2000\, e^{b(5)}$

$\Rightarrow\ 3 = e^{5b} \ \Rightarrow\ \ln 3 = 5b \ \Rightarrow\ b = \ln 3/5.$

Thus, $y(t) = 2000 e^{\left(\frac{\ln 3}{5}\right)t}$, with $\boxed{a = 2000}$ and $\boxed{b = \frac{\ln 3}{5}}$.

Example 16: A bacteria culture starts with 5,000 bacteria and the population quadruples after 3 hours. Find an expression for the number $P(t)$ of bacteria after t hours.

(times 4)

Solve for b: $4 = e^{b3} \Rightarrow \ln 4 = 3b \Rightarrow b = \frac{\ln 4}{3}$

$$\boxed{P(t) = 5000\, e^{\frac{\ln 4}{3} t}}$$

(can begin with any number of cells, say 1, and after 3 hours there are 4. If you start with more, equation still simplifies to this form.)

Example 17: If the bacteria in a culture doubles in 3 hours, how many hours will it take before 7 times the original number is present?

Step 1: find $P(t)$.

$\hookrightarrow$ Solve for b: $2 = e^{b \cdot 3} \Rightarrow \ln 2 = 3b \Rightarrow b = \frac{\ln 2}{3}$

$P(t) = P_0 e^{(\ln 2/3)t}$

Step 2: Solve for t: If we begin with P_0, how long until we have $7P_0$?

$7P_0 = P_0 e^{(\ln 2/3)t} \Rightarrow 7 = e^{\left(\frac{\ln 2}{3}\right)t} \Rightarrow \ln 7 = \left(\frac{\ln 2}{3}\right)t$

divide by P_0

$\Rightarrow \boxed{t = \ln 7 \cdot \frac{3}{\ln 2} \text{ hours}} \approx 8.422 \text{ hours}$

Example 18: If the world population in 2010 was 6 billion and it were to grow exponentially with a growth constant $r = \frac{1}{30}\ln(2)$, find the population (in billions) in the year 2070.

Let $P(t)$ be the population t years after 2010. (in billions)

So $P(t) = 6e^{\frac{\ln 2}{30}t}$. The year 2070 is 60 years after 2010,

So we want $P(60) = \boxed{6e^{\frac{\ln 2}{30}(60)}} = \boxed{6e^{2\ln 2}} = 6e^{\ln 4}$

log property

$= 6(4) = \boxed{24 \text{ billion people}}$

Radioactive decay:

Radioactive substances decay by spontaneously emitting radiation. In this situation, the rate of decay is proportional to the mass of the substance.

This is analogous to population growth, except that the quantity of radioactive material *decreases*.

Remark: Physicists sometimes express the rate of decay in terms of the **half-life**, the time required for half the mass to decay.

Radioactive decay model:

If Q_0 is the initial quantity of a radioactive substance with half-life t_0, then the quantity $Q(t)$ remaining at time t is modeled by the function

$$\boxed{Q(t) = Q_0 e^{-rt}}$$

where $r = \dfrac{\ln(2)}{t_0}$.

Note: If t_0 denotes the half-life of a radioactive substance, we can rewrite the expression for $Q(t)$ as follows

$$Q(t) = Q_0\, e^{-rt} = Q_0\, e^{-(\ln(2)/t_0)\cdot t} = Q_0 \left(e^{\ln(2)}\right)^{-t/t_0} = Q_0\, 2^{-t/t_0} = Q_0\,(2^{-1})^{t/t_0} = Q_0 \left(\frac{1}{2}\right)^{t/t_0}.$$

Example 19: The half-life of Cesium-137 is 30 years. Suppose we have a 100 gram sample. How much of the sample will remain after 50 years?

$\hookrightarrow$ Solve for b: If we start with 1 gram, after 30 years we have $\frac{1}{2}$ gram: $\frac{1}{2} = e^{b(30)} \Rightarrow \ln\frac{1}{2} = 30b$

$\Rightarrow b = \frac{\ln\frac{1}{2}}{30}$. Thus, $P(t) = 100\, e^{\left(\frac{\ln\frac{1}{2}}{30}\right)t}$

After 50 years, $P(50) = \boxed{100\, e^{\left(\frac{\ln\frac{1}{2}}{30}\right)\cdot 50} \text{ grams}} \approx 31.5 \text{ grams}$