MA123, Chapter 7: Rates of Change in the Natural and Social Sciences

Chapter Goals:
- Understand position, velocity, and acceleration function.
- Understand derivatives as rates of change in more general applications.
- Understand marginal functions in economics.

Assignments: Assignment 13

Rates of Change in Motion

recall: speed is absolute value of $v(t)$

Suppose the position of an object at time t is given by the function $s(t)$. We learned in Chapter 2 that we can measure the average velocity over an interval of time, or the instantaneous velocity, $v(t)$, by computing the derivative of $s(t)$. Similarly, we can consider the rate that velocity changes by computing $v'(t)$; this is called the *acceleration*.

Rate of Change in Motion:

Position	$s(t)$
Velocity	$v(t) = s'(t)$
Acceleration	$a(t) = v'(t) = s''(t)$

We can pay attention to the units: for example, if $s(t)$ is measured in meters and time is measured in seconds, then the units for $v(t)$ are m/s and the units for $a(t)$ are m/s^2.

Example 1: The function $s(t)$ describes the position of a particle moving along a coordinate line, where s is in feet and t is in seconds.

$$s(t) = \frac{1}{80}t^2 - \ln(t+3), \qquad t \geq 0$$

a) Find the velocity function $v(t)$ and the acceleration function $a(t)$.

$$v(t) = s'(t) = \frac{1}{80}(2)t - \frac{1}{t+3} = \boxed{\frac{1}{40}t - \frac{1}{t+3}} \quad \text{or} \quad = \frac{1}{40}t - (t+3)^{-1}$$

$$a(t) = v'(t) = s''(t) = \frac{1}{40} + (t+3)^{-2} = \boxed{\frac{1}{40} + \frac{1}{(t+3)^2}}$$

b) Find the position, velocity, speed, and acceleration at $t = 1$.

position: $S(1) = \frac{1}{80}(1)^2 - \ln(1+3) = \boxed{\frac{1}{80} - \ln 4 \text{ feet}}$

$v(1) = \frac{-9}{40}$ ft/sec

speed $= |v(1)| = \left|\frac{1(1)}{40} - \frac{1}{1+3}\right| = \left|\frac{1}{40} - \frac{10}{40}\right| = \left|-\frac{9}{40}\right| = \boxed{\frac{9}{40} \text{ ft/sec}}$

acceleration $= a(1) = \frac{1}{40} + \frac{1}{(1+3)^2} = \frac{1}{40} + \frac{1}{16} = \frac{2}{80} + \frac{5}{80} = \boxed{\frac{7}{80} \text{ ft/sec}^2}$

c) At what times is the particle stopped?

particle stopped when $v(t) = 0$:

$$\frac{1}{40}t - \frac{1}{t+3} = 0$$

$$\frac{t}{40} = \frac{1}{t+3}$$

cross-multiply

$$t(t+3) = 40$$

$$t^2 + 3t - 40 = 0$$

$$(t+8)(t-5) = 0$$

$$t = -8, \quad \boxed{t = 5}$$

$t = -8$ is not in the domain of the original $S(t)$.

particle stopped at $t = 5$ seconds.

Other Applications In general, consider the function $y = f(x)$. The derivative $f'(x)$ is a function that measures the **rate** y **changes with respect to** x. The units for $f'(x)$ can be expressed in fraction form, as $\frac{\text{units of } y}{\text{units of } x}$.

Example 2: The height of a sand dune (in centimeters) is represented by $f(t) = 650 - 9t^2$ cm, where t is measured in years since 1991. Find $f(8)$ and $f'(8)$ and determine what each means in terms of the sand dune.

$f(8) = 650 - 9(8)^2 = $ 74 cm, the height of the sand dune in 1999.

$f'(t) = -18t$, so $f'(8) = -18(8) = -144$ cm/year:

in 1999, the dune is shrinking at a rate of 144 cm/year.

Example 3: If a tank holds 4000 gallons of water, which drains from the bottom ofthe tank in 50 minutes, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V(t) = 4000\left(1 - \frac{t}{50}\right)^2 \qquad 0 \leq t \leq 50.$$

Find the rate at which the water is draining out of the tank after 15 minutes:

$$V'(t) = \underbrace{4000(2)\left(1-\frac{t}{50}\right)}_{\text{power rule}}\underbrace{\left(\frac{-1}{50}\right)}_{\text{chain rule}} = -160\left(1-\frac{t}{50}\right)$$

$$V'(15) = -160\left(1-\frac{15}{50}\right) = -160\left(\frac{35}{50}\right) = \boxed{-112 \text{ gallons/minute}}$$

Example 4: One hour after x milligrams of a drug are given to a person, the change in body temperature $T(x)$ (in degrees Fahrenheit) is given by the function

$$T(x) = x^2\left(1 - \frac{x}{2}\right).$$

Further, the rate $T'(x)$ at which T changes with respect to the size of the dosage, x, is called the **sensitivity** of the body to the dosage. Find the sensitivity of the body when the dosage is 3 milligrams.

We can use product rule, or distribute to rewrite $T(x) = x^2 - \frac{1}{2}x^3$.

Then $T'(x) = 2x - \frac{3}{2}x^2$

The sensitivity at a dosage of 3 mg is $T'(3) = 2(3) - \frac{3}{2}(3)^2$

$$= 6 - \frac{27}{2} = \frac{-15}{2} = \boxed{-7.5 \text{ degrees/mg}}$$

Example 5: A study of American girls ages 4-13 in the 90^{th} percentile found that their height h (in centimeters) as a function of their age a (in years) satisfies the equation

$$h(a) = 6.42a + 82.2.$$

The same study found that their weight W (in grams) as a function of their height is given by

$$W(h) = 0.0302h^{2.82}.$$

a) What is the growth rate of height as a function of time?

$$h'(a) = 6.42 \ \text{cm}/\text{year}$$

b) Write an expression for the composition function that gives the weight as a function of age.

$$W(a) = 0.0302(6.42a + 82.2)^{2.82}$$

c) Differentiate this function to find $W'(a)$ using the chain rule. What is the rate of change in weight at age 11?

$$W'(a) = \underbrace{0.0302(2.82)(6.42a + 82.2)^{1.82}}_{\text{power rule}} \underbrace{(6.42)}_{\text{chain rule}}$$

$$= .54675(6.42a + 82.2)^{1.82}$$

$$W'(11) = .54675(6.42(11) + 82.2)^{1.82} = .54675(152.82)^{1.82} \approx \boxed{5164 \ \text{gm}/\text{year}} \ *$$

(* WeBWorK will want more decimal places)

Applications in Business and Economics

In economics, we are concerned with cost, revenue, and profit functions, based on the number of goods produced or sold. For example, suppose we wish to manufacture x widgets. We expect some ***fixed costs*** such as equipment and other infrastructure. We also expect ***variable costs***, such as the cost of the material needed to produce each widget. For example, if the fixed costs were \$3000 and each widget requires \$15 worth of material, our cost function is

$$C(x) = 3000 + 15x.$$

If we take other factors into account, we can get more complicated cost functions.

When we sell items, the money we receive is called the *revenue*. The quantity we expect to sell is related to the price that we charge. For example, if the price-demand function is $p = 170 - 0.6x$, then the revenue function, which is the price of each item times the number of items sold, is

$$R(x) = (170 - 0.6x) \cdot x = 170x - 0.6x^2.$$

Our ***profit*** in selling the widgets is the difference between the amount we spent to make them and the amount we receive from selling them, so $P(x) = R(x) - C(x)$. In our example,

$$P(x) = 170x - 0.6x^2 - (3000 + 15x) = -0.6x^2 + 155x - 3000.$$

At a production level of 50 widgets, our cost is $C(50) = \$3750$, our revenue from selling the 50 widgets is $R(50) = \$7000$, and our profit is the difference, $P(50) = \$3250$.

Marginal Functions

The ***marginal cost function*** is $C'(x)$, the derivative of the cost function. When we evaluate the marginal cost function for a particular value of x, we approximate the cost of producing one more unit at that production level. Similarly, we can find the marginal revenue function $R'(x)$ or the marginal profit function $P'(x)$. For our example above, we have a marginal profit function

$$P'(x) = -1.2x + 155.$$

At a production level of 50 units, the marginal profit is $P'(50) = \$95$, so we estimate that the profit from producing and selling the 51^{st} unit is about \$95. The *exact* profit in producing and selling the 51^{st} unit is

$$P(51) - P(50) = 3344.40 - 3250 = \$94.40,$$

which shows that the \$95 is a good estimate.

Example 6: The price-demand and cost functions for the production of microwaves are given as

$$p = 270 - \frac{x}{50}$$

and

$$C(x) = 66000 + 100x,$$

where x is the number of microwaves that can be sold at a price of p dollars per unit and $C(x)$ is the total cost (in dollars) of producing x units.

a) Find the marginal cost as a function of x.

$$C'(x) = 100$$

(in this example, marginal cost is \$100 for every x).

b) Find the revenue function in terms of x.

$$R(x) = x \cdot p = x\left(270 - \frac{x}{50}\right) = 270x - \frac{1}{50}x^2$$

c) Find the marginal revenue function in terms of x.

$$R'(x) = 270 - \frac{1}{50}(2x) = 270 - \frac{x}{25}$$

d) Evaluate the marginal revenue function at $x = 1500$.

$$R'(1500) = 270 - \frac{1500}{25} = \$210$$

at a production level of 1500 microwaves, we expect the 1501^{st} microwave to bring in ~\$210 more dollars

e) Find the profit function in terms of x.

$$P(x) = R(x) - C(x) = 270x - \frac{1}{50}x^2 - (66000 + 100x)$$
$$= 170x - \frac{1}{50}x^2 - 66000$$

f) Evaluate the marginal profit function at $x = 1500$.

$$P'(x) = 170 - \frac{2}{50}x - 0 = 170 - \frac{x}{25}$$
$$P'(1500) = 170 - \frac{1500}{25} = \boxed{\$110}$$

← We expect the 1501st microwave to net about \$110 profit

Average Cost

Given a cost function $C(x)$, we can also find the ***average cost function*** $\bar{C}(x)$ by dividing by x.

Given this, we can compute the ***marginal average cost function*** $\bar{C}'(x)$, which can be used to estimate how much the average cost will change if we produce one more unit. We can do the same for the revenue and profit functions as well.

Average Cost Function:

If $C(x)$ is the cost of producing x units, then the average cost of producing x units is the function

$$\boxed{\bar{C}(x) = \frac{C(x)}{x}}$$

Example 7: The total profit (in dollars) from the sale of x charcoal grills is

$$P(x) = 20x - 0.1x^2 - 225.$$

a) Find the average profit per grill if 30 grills are produced.

$$\bar{P}(x) = \frac{P(x)}{x} = \frac{20x - 0.1x^2 - 225}{x} = 20 - 0.1x - 225x^{-1}$$
$$\bar{P}(30) = 20 - 0.1(30) - \frac{225}{30} = \boxed{\$9.50}$$

← or: $P(30) = \$285$, so $\frac{P(30)}{30} = \$9.50$

b) Find the marginal average profit at a production level of 30 grills.

$$\bar{P}'(x) = -0.1 + 225x^{-2} = -0.1 + \frac{225}{x^2}$$
$$\bar{P}'(30) = -0.1 + \frac{225}{30^2} = \boxed{\$0.15}$$

At a production level of 30 grills, making one more grill will increase the avg. profit by about 15 ¢.

c) Use the results from parts a) and b) to estimate the average profit if 31 grills are produced.

$$\$9.50 + 0.15 = \boxed{\$9.65}$$

↑ avg. profit per grill at 30 grills

↑ increase per grill if we make more