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MA 123

Introductory Worksheet #1: Average Rate of Change

Calculus describes how quantities change. This worksheet introduces *average rate of change*.

Motivating Questions

1. Given a function $f(x)$, how do we calculate the average rate of change of f between $x = x_1$ and $x = x_2$?

Let's start with a concept that you are already familiar with: speed. The average speed of an object is equal to the ratio of the distance traveled by that object over the time elapsed:

$$\text{average speed} = \frac{\text{distance traveled}}{\text{time elapsed}}.$$

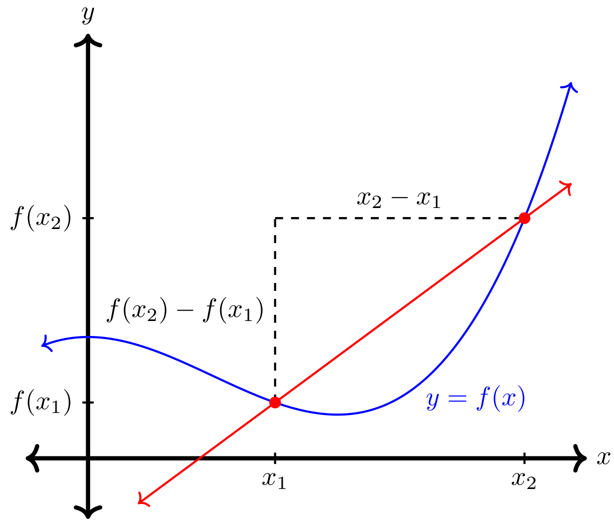
Practice 1. A train leaves city A at 8:00am and arrives at City B at 1:30pm. The distance from city A to city B is 220 miles. What was the average speed of the train in miles per hour (mph)?

Practice 2. A train leaves city A at 10:00am and arrives at city B at 12:00pm. The train stops at city B for 1 hour and then continues to city C. It arrives at city C at 4:00pm. The distance from city A to city B is 60 miles, and the average speed from city B to city C was 40 mph. What was the average speed of the train from city A to city C (including stopping time)? *Hint: It may help to sketch a picture.*

When computing average rates of change of a quantity y with respect to a quantity x, there is generally a function f that shows how the values x and y are related. The **average rate of change of the function $y = f(x)$ between $x = x_1$ and $x = x_2$** is given by

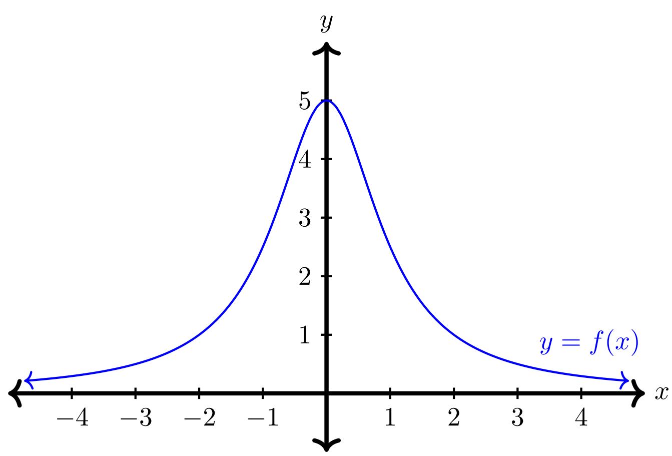
$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}. \tag{1}$$

As shown below, the average rate of change is the *slope of the secant line between $x = x_1$ and $x = x_2$ on the graph of $y = f(x)$*.



$$\text{slope of secant line} = \frac{\text{rise}}{\text{run}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Practice 3. Below is the graph of a function $f(x)$. Use the graph to determine the average rate of change of f from $x = 0$ to $x = 2$.



Practice 4. The graph in Practice 3 is of the function $f(x) = \dfrac{5}{x^2+1}$. Use Equation (1) to determine the average rate of change of f from $x = 0$ to $x = 2$. How does this answer compare with your answer from Practice 3?

Practice 5. Let $g(x) = x^2 - 5x + 3$.

(a) Determine $g(-4)$.

(b) Determine $g(-4 + h)$.

(c) Determine the average rate of change of g on the interval $[-4, -4 + h]$.

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Introductory Worksheet #2: Limits and Continuity

This worksheet introduces *limits* (*continuity* will be introduced in Review Worksheet #2).

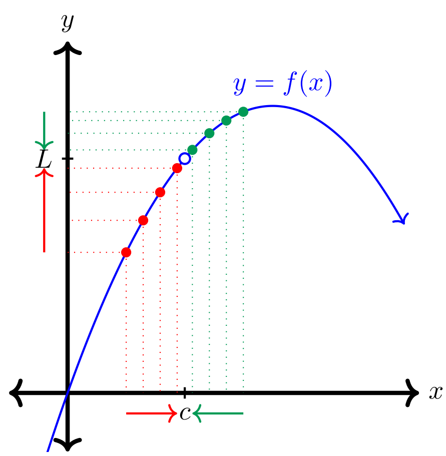
Motivating Questions

1. Given a function $f(x)$, how do we calculate the limit of $f(x)$ as x approaches c?

Computing a limit means computing what happens to the value of a function as the independent variable gets closer and closer to (but does not equal) a particular value. Let f be a function of x. The notation

$$\lim_{x\to c} f(x) = L$$

means that the values of $f(x)$ get closer and closer to the value L as x gets closer and closer to c (see the graph below).



As we see from the graph, the y-values are approaching L as the x-values approach c from the left. Likewise the y-values are approaching L as the x-values approach c from the right. Then $\lim_{x\to c} f(x) = L$ since the values of $y = f(x)$ are approaching L as x gets closer and closer to c from either side (note that the function is undefined at $x = c$).

As illustrated above, limits can be one-sided.

- The limit of $f(x)$ as x approaches c from the left (called a *left-sided limit*) is denoted $\lim_{x\to c^-} f(x)$. This is the limit of $f(x)$ as x gets closer to c for values of x less than c.

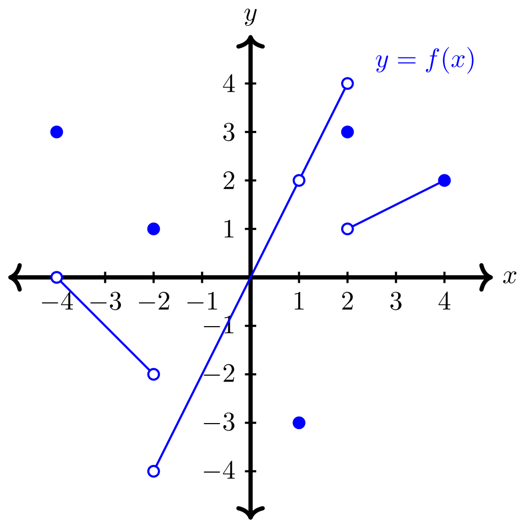
- The limit of $f(x)$ as x approaches c from the right (called a *right-sided limit*) is denoted $\lim_{x\to c^+} f(x)$. This is the limit of $f(x)$ as x gets closer to c for values of x greater than c.

Then, we say that a limit $\lim_{x\to c} f(x)$ exists if and only if

1. $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist, and

2. $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = \lim_{x\to c} f(x)$.

Practice 6. Below is the graph of a piecewise function $f(x)$.



Use the graph to evaluate the following limits/values.

(a) $\lim_{x \to -2^-} f(x)$

(b) $\lim_{x \to -2^+} f(x)$

(c) $\lim_{x \to -2} f(x)$

(d) $f(-2)$

(e) $\lim_{x \to 0^-} f(x)$

(f) $\lim_{x \to 0^+} f(x)$

(g) $\lim_{x \to 0} f(x)$

(h) $f(0)$

(i) $\lim_{x \to 1^-} f(x)$

(j) $\lim_{x \to 1^+} f(x)$

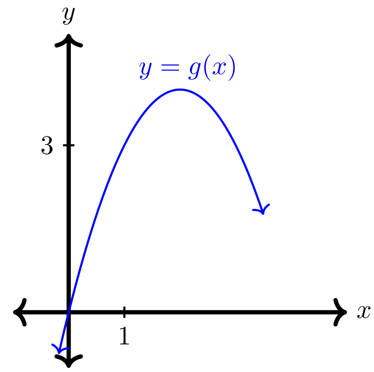
(k) $\lim_{x \to 1} f(x)$

(l) $f(1)$

Of course, we don't want to always have to graph a function in order to evaluate its limit near a specified value. The next practice problem shows that for "nice enough" functions (which will be discussed in Review Worksheet #2), we can evaluate the function at the specified value.

Practice 7. Let $g(x) = -x^2 + 4x$.

(a) Below is the graph of $y = g(x)$ near $x = 1$. Use the graph to determine $\lim_{x \to 1} g(x)$.



(b) Fill in the table of values for $g(x)$. What happens to the value of $g(x)$ as the values of x get closer and closer to $x = 1$ from both the left and right?

x	0.5	0.7	0.9	0.99	0.999
$g(x)$					

x	1.5	1.3	1.1	1.01	1.001
$g(x)$					

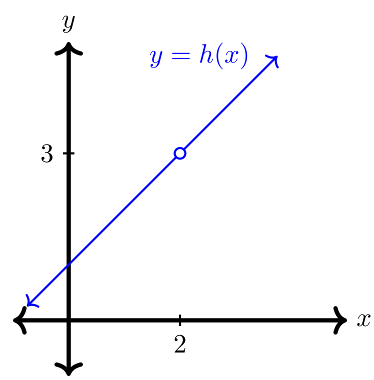
(c) Evaluate $\lim_{x \to 1} g(x)$ by substituting $x = 1$ into the function $g(x) = -x^2 + 4x$. How does this result compare to the results from parts (a) and (b)?

Sometimes, we'll have to do a bit of extra work before direct substitution yields the desired limit.

Practice 8. Evaluate $\lim_{x\to 2} \dfrac{x^2 - x - 2}{x - 2}$.

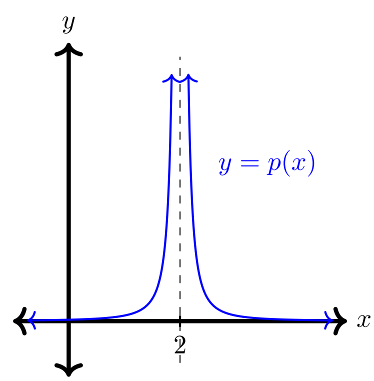
(a) Try substituting $x = 2$ into the function $h(x) = \dfrac{x^2 - x - 2}{x - 2}$. What issues arise?

(b) Factor the numerator of $h(x) = \dfrac{x^2 - x - 2}{x - 2}$ before substituting $x = 2$. Does this result seem correct given the graph of $y = h(x)$ below?



Limits can also fail to exist. When this occurs, we say the limit does not exist or DNE.

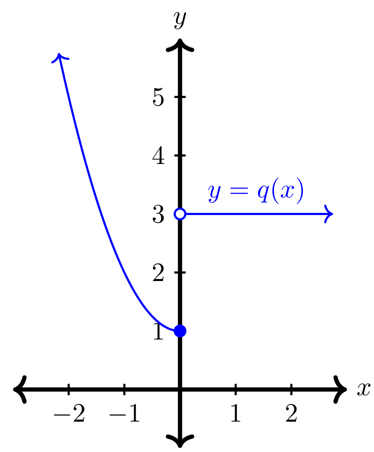
Practice 9. Evaluate $\lim_{x\to 2} \dfrac{1}{(x-2)^2}$ by substituting $x = 2$ into the function $p(x) = \dfrac{1}{(x-2)^2}$. What issues arise? Will factoring help resolve any issues? The graph of $y = p(x)$ is provided below.



Practice 9 raises the question of when limits fail to exist. There are two ways limits can fail to exist:

- The function grows without bound as $x \to c$. See Practice 9 for an example of this.
- The function approaches multiple values as $x \to c$. See Practice 10 for an example of this.

Practice 10. Evaluate $\lim_{x \to 0} q(x)$, where $q(x) = \begin{cases} x^2 + 1 & \text{if } x \leq 0, \\ 3 & \text{if } x > 0, \end{cases}$ by considering $\lim_{x \to 0^-} q(x)$ and $\lim_{x \to 0^+} q(x)$. Do both limits exist? Are the limits equal? You may use the graph of $y = q(x)$ provided below.



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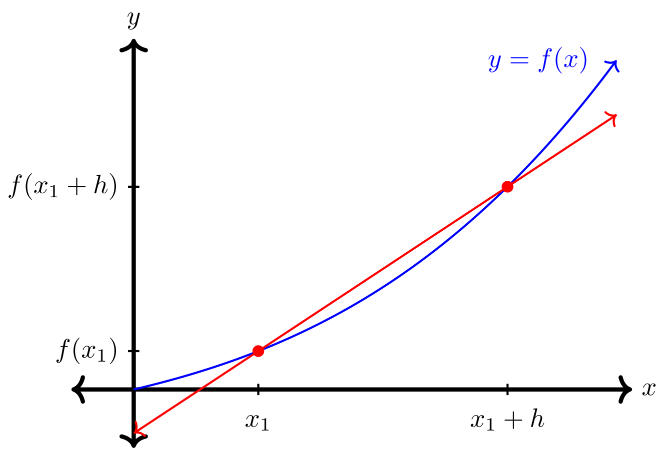
Introductory Worksheet #3: Instantaneous Rate of Change and The Derivative

This worksheet introduces *instantaneous rate of change* and *derivatives*.

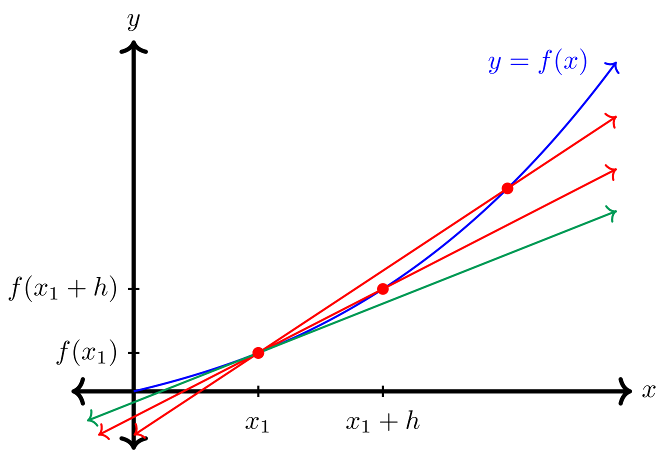
Motivating Questions

1. Given a function $f(x)$, how do we calculate the instantaneous rate of change of f at $x = x_1$?
2. Given a function $f(x)$, how do we determine the derivative of $f(x)$?

Going back to the familiar concept of speed, how can we determine the speed of an object at a specific time? That is, how can we determine an object's instantaneous speed? Suppose the graph of $y = f(x)$ below gives the total distance traveled by an object at time x.



The slope of the red secant line gives the *average speed* of the object over the interval $[x_1, x_1 + h]$. How can we use this information to help us determine the *instantaneous speed* of the object at time $x = x_1$? By decreasing the value of h, we can determine the average speed of the object over a shorter interval $[x_1, x_1 + h]$. Then, the instantaneous speed is obtained by letting $h \to 0$.



The slope of the green line yields the instantaneous speed of the object at time $x = x_1$. In general, the **instantaneous rate of change of a function $f(x)$ at $x = x_1$** is given by

$$\text{instantaneous rate of change} = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}. \tag{2}$$

As shown in the graph at the bottom of the previous page, the instantaneous rate of change is the *slope of the tangent line to the graph of $y = f(x)$ at $x = x_1$*. [Note: A tangent line is a line that touches a curve at a point but does not cross the curve at that point.]

Practice 11. Let $f(x) = x^2 - 5x + 3$.

(a) Determine $f(-4 + h)$.

(b) Determine $f(-4 + h) - f(-4)$.

(c) Determine $\dfrac{f(-4 + h) - f(-4)}{h}$.

(d) Use Equation (2) with the result from part (c) to determine the instantaneous rate of change of $f(x)$ at $x = -4$.

The instantaneous rate of change of a function $f(x)$ at a general value x is called the **derivative of f at x**, and is denoted $f'(x)$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{3}$$

Geometrically, the derivative of $f(x)$ at $x = x_1$ is the slope of the tangent line to the graph of f at $x = x_1$.

Practice 12. Let $f(x) = x^2 - 5x + 3$.

(a) Determine $f(x+h)$.

(b) Determine $f(x+h) - f(x)$.

(c) Determine $\dfrac{f(x+h) - f(x)}{h}$.

(d) Use Equation (3) with the result from part (c) to determine the derivative of $f(x)$.

(e) Substitute $x = -4$ in the result from part (d). How does this result compare with the value from Practice 11(d)?

Practice 13. Use Equation (3) to determine the derivative of $g(x) = (x+4)^2$. *Hint: It may help to expand* $(x+4)^2$ *in the form* $ax^2 + bx + c$.

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Introductory Worksheet #4: Tangent Lines and Differentiability

This worksheet introduces *equations of tangent lines* and *differentiability*.

Motivating Questions

1. Given a function $f(x)$, how do we determine an equation of a tangent line to $f(x)$ at $x = x_0$?
2. Given a function $f(x)$, how do we determine where the function is differentiable?

Recall that the *slope-intercept form* of an equation of a line is

$$y = mx + b,$$

where m is the slope of the line and b is the y-coordinate of the y-intercept. The *point-slope form* of an equation of a line is

$$y - y_0 = m(x - x_0),$$

where m is the slope of the line and (x_0, y_0) is a point on the line.

Let $f(x)$ be a function. Note the following:

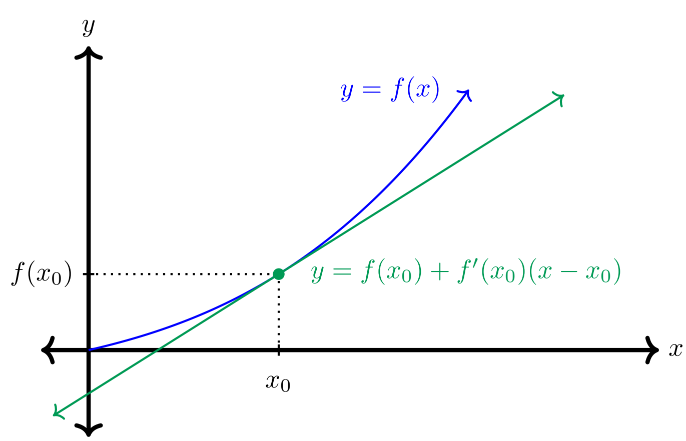
1. The point $(x_0, f(x_0))$ is a point on the graph of $y = f(x)$.
2. **N.B.**[1] The slope of the tangent line to the graph of $y = f(x)$ at $x = x_0$ is the instantaneous rate of change of the function f at $x = x_0$, which is the derivative of f at $x = x_0$. Thus, the slope of the tangent line to the graph of $y = f(x)$ at $x = x_0$ is $m = f'(x_0)$.

Therefore, an equation of the tangent line to the graph of $y = f(x)$ at $x = x_0$ is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

or

$$y = f(x_0) + f'(x_0)(x - x_0).$$



[1]Nota Bene: Latin phrase meaning "note well".

Practice 14. Let $f(x) = x^2 - 5x + 3$. Determine the slope-intercept form of the equation of the tangent line to the graph of $y = f(x)$ at $x = -4$.

(a) What is the value of x_0?

(b) Evaluate $f(x_0)$.

(c) Evaluate $f'(x_0)$.

(d) Use the previous parts to determine an equation of the tangent line in point-slope form.

(e) Convert the point-slope form of the equation of the tangent line to slope-intercept form. What is the value of the slope, m? What is the value of the y-coordinate of the y-intercept, b?

Practice 15. Let $g(x) = (x + 4)^2$. Determine the slope-intercept form of the equation of the tangent line to the graph of $y = g(x)$ at $x = 1$.

Recall from Review Worksheet #2 that a function is continuous at a point provided the graph of the function does not have a hole, a jump, or a vertical asymptote at that point. How can we tell if a function is differentiable at a point by looking at its graph?

To answer this question, we first have to know what it means for a function to be differentiable at a point. A function f is said to be **differentiable at $x = c$** if the derivative of f exists at $x = c$. That is, f is differentiable at $x = c$ if the limit

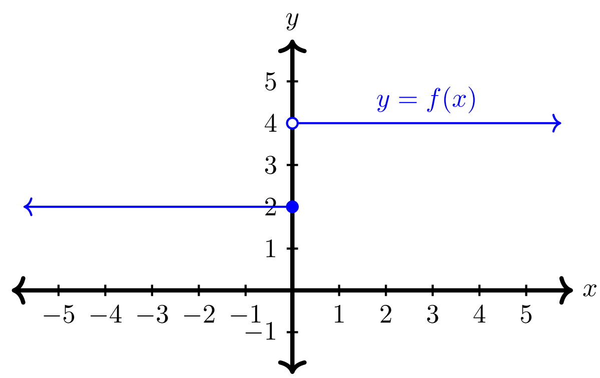
$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists. The next problem considers two functions that are not differentiable at $x = 0$.

Practice 16. Show that each of the following functions are not differentiable at $x = 0$.

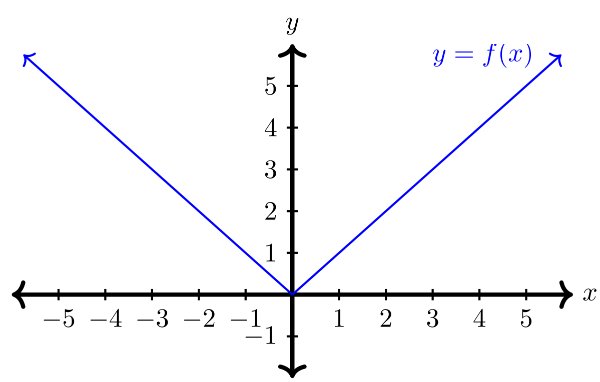
(a) $f(x) = \begin{cases} 2 & \text{if } x \leq 0, \\ 4 & \text{if } x > 0 \end{cases}$

Hint: Evaluate $\displaystyle\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$.



(b) $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0 \end{cases}$

Hint: Evaluate $\displaystyle\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h}$ *and* $\displaystyle\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$. *Do they match?*



As we showed in Practice 16, a function is not differentiable wherever the graph of the function has a jump or a *corner point*. We will show in Review Worksheet #4 that a function is not differentiable wherever the graph of the function has a hole or a vertical asymptote. Then, a function is not differentiable wherever one of the following occurs:

1. The graph of the function has a hole.
2. The graph of the function has a jump.
3. The graph of the function has a vertical asymptote.
4. The graph of the function has a corner point.

Note that if a function is not continuous at a point, then it is also not differentiable at that point. However, the converse is not always true (see Practice 16(b) – the function $f(x) = |x|$ is not differentiable at $x = 0$ but it is continuous at $x = 0$).

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Introductory Worksheet #5: Formulas for Derivatives

This worksheet introduces the *constant*, *power*, *product*, *constant multiple*, and *sum/difference* rules for derivatives. The *quotient* and *chain* rules for derivatives will be introduced in Review Worksheet #5.

Motivating Questions

1. Given a function $f(x)$, how do we compute its derivative using various rules?

Computing a derivative using the limit definition is time consuming. Thankfully there are formulas for determining the derivative of certain types of functions that allow us to compute the derivative without needing to use the limit definition of the derivative.

The Constant Rule

The derivative of a constant function $f(x) = c$ is $f'(x) = 0$.

Practice 17. Let $f(x) = -3$. Use the constant rule to determine $f'(x)$.

The Power Rule

The derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$.

Practice 18. Use the power rule to determine the derivative of each of the following functions.

(a) $f(x) = x^3$

(b) $f(x) = \sqrt{x}$
Hint: Recall $\sqrt[n]{x} = x^{1/n}$.

(c) $f(x) = \dfrac{1}{x^4}$
Hint: Recall $\dfrac{1}{x^n} = x^{-n}$.

The Constant Multiple Rule

Let c be a constant and $f(x)$ be a differentiable function. Then $(cf(x))' = cf'(x)$.

Practice 19. Determine the derivative of each of the following functions using the constant multiple rule.

(a) $g(x) = 4x^7$

(b) $g(x) = \dfrac{1}{2x^3}$

The Sum/Difference Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then $(f(x) \pm g(x))' = f'(x) \pm g'(x)$.

Practice 20. Determine the derivative of each of the following functions using the sum/difference rule.

(a) $h(x) = x^8 + \dfrac{1}{x^8}$

(b) $h(x) = -6x^5 + 3x^2 - \sqrt{x} + 14$

The Product Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.

Practice 21. Determine the derivative of $h(x) = (7x + 2)(2x^3 - 5x)$.

(a) Identify two functions $f(x)$ and $g(x)$ such that $h(x) = f(x)g(x)$.

(b) Determine $f'(x)$ and $g'(x)$ using the functions from part (a).

(c) Determine $h'(x)$ using the product rule and results from the previous parts.

(d) Note that $(7x + 2)(2x^3 - 5x) = 14x^4 + 4x^3 - 35x^2 - 10x$. Determine the derivative of $14x^4 + 4x^3 - 35x^2 - 10x$ using the power and sum/difference rules, and compare this result with part (c). *Hint: You may need to simplify the result from part (c).*

(e) Compute $f'(x) \cdot g'(x)$, and compare this result with part (c). Does $(f(x)g(x))' = f'(x) \cdot g'(x)$?

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Introductory Worksheet #6: Exponential and Logarithmic Functions

This worksheet introduces derivatives of *exponential* and *logarithmic* functions.

Motivating Questions

1. How do we compute the derivative of an exponential function?
2. How do we compute the derivative of a logarithmic function?

An exponential function is of the form $f(x) = a^x$ where $a > 0$ and $a \neq 1$. Logarithmic functions are of the form $g(x) = \log_a(x)$ where $a > 0$ and $a \neq 1$. Two important properties of exponential and logarithmic functions are that

1. $\log_a(a^x) = x$ and
2. $a^{\log_a(x)} = x$.

That is, $f(x) = a^x$ and $g(x) = \log_a(x)$ are inverse functions.

If we choose our base to be $a = e \approx 2.71828$, then we get the natural exponential function $f(x) = e^x$ and the natural logarithmic function $g(x) = \log_e(x) \stackrel{\text{def}}{=} \ln(x)$. We will only work with base $a = e$ in this class.

The Natural Exponential Function

The derivative of $f(x) = e^x$ is $f'(x) = e^x$.

Practice 22. Determine the derivative of each of the following functions.

(a) $f(x) = e^{2x}$
Hint: Use the chain rule.

(b) $f(x) = 5e^{3x^2+7}$
Hint: Use the chain and constant multiple rules.

(c) $f(x) = xe^x$
Hint: Use the product rule.

In general, $\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$, as shown below using the chain rule. Applying the chain rule to $e^{f(x)}$:

$$\text{Inner: } f(x) \xrightarrow{d/dx} \boxed{f'(x)}$$

$$\text{Outer: } e^x \xrightarrow{d/dx} e^x \xrightarrow{\text{inner}} \boxed{e^{f(x)}}$$

Multiplying the two boxed results yields $\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$.

The Natural Logarithmic Function

The derivative of $g(x) = \ln(x)$ is $g'(x) = \frac{1}{x}$.

Practice 23. Determine the derivative of each of the following functions.

(a) $g(x) = \ln(2x)$
Hint: Use the chain rule.

(b) $g(x) = 5\ln(3x^2 + 7)$
Hint: Use the chain and constant multiple rules.

(c) $g(x) = x\ln(x)$
Hint: Use the product rule.

In general, $\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$, as shown below using the chain rule. Applying the chain rule to $\ln(f(x))$:

$$\text{Inner: } f(x) \xrightarrow{d/dx} \boxed{f'(x)}$$

$$\text{Outer: } \ln(x) \xrightarrow{d/dx} \frac{1}{x} \xrightarrow{\text{inner}} \boxed{\frac{1}{f(x)}}$$

Multiplying the two boxed results yields $\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$.

The last two problems involve functions where we only know some information about those functions (but not the functions themselves).

Practice 24. Suppose $h(x) = f(x)e^{g(x)}$. If $f(-1) = 2$, $f'(-1) = 5$, $g(-1) = 1$, and $g'(-1) = -7$, determine $h'(-1)$. *Hint: Use the product rule.*

Practice 25. Suppose $h(x) = f(\ln(x)) + \ln(g(x))$. If $f'(0) = 9$, $g(1) = -4$, and $g'(1) = 12$, determine $h'(1)$. *Hint: Use the chain rule.*

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Introductory Worksheet #7: Applications of the Derivative

This worksheet explores *applications of the derivative*.

Motivating Questions

1. How are derivatives used in the real world?

Practice 26. A coal-burning plant emits sulfur dioxide into the surrounding air. The concentration $C(x)$, in parts per million, is given by the function

$$C(x) = \frac{0.5}{x^2},$$

where x is the distance away from the plant in miles. Determine the instantaneous rate of change of the sulfur dioxide concentration 4 miles from the plant (include units).

Practice 27. A drug is injected into the bloodstream of a patient. The concentration of the drug in the bloodstream (in mg per cm^3) t hours after the injection is given by

$$C(t) = \frac{0.16t}{t^2 + 4}.$$

Determine the instantaneous rate of change of the drug concentration after 30 minutes (include units).

An application in economics relates to *cost*, *revenue*, and *profit* functions, which are based on the number of goods produced or sold. The **cost function** is the sum of two components, *fixed* costs and *variable* costs. For instance, equipment and infrastructure are fixed costs. These costs do not depend on the number of goods produced. On the other hand, material is a variable cost as the amount of material needed for production depends on the number of goods produced. We will use the notation $C(x)$ to denote the total cost of producing x items.

When a business sells items, the money that it receives is called revenue. The **revenue function** is equal to the price of each item times the number of items sold. We will use the notation $R(x)$ to denote the total revenue from selling x items.

The profit made by a business is the difference between revenue and cost. That is, the **profit function** is $P(x) = R(x) - C(x)$, where $P(x)$ is the profit from producing/selling x items.

How do derivatives come into play? We can consider what is called the marginal cost, marginal revenue, or marginal profit. The **marginal cost function** $C'(x)$ *approximates* the additional cost of producing one more item when x items are currently being produced. The **marginal revenue function** $R'(x)$ *approximates* the additional revenue from selling one more item when x items are currently being sold. The **marginal profit function** $P'(x)$ *approximates* the additional profit from producing/selling one more item when x items are currently being produced/sold.

We can also consider average cost, average revenue, average profit, marginal average cost, marginal average revenue, and marginal average profit. The **average cost function** $\bar{C}(x)$ is

$$\bar{C}(x) = \frac{C(x)}{x}.$$

Likewise, the **average revenue function** $\bar{R}(x)$ is

$$\bar{R}(x) = \frac{R(x)}{x}$$

and the **average profit function** $\bar{P}(x)$ is

$$\bar{P}(x) = \frac{P(x)}{x}.$$

The **marginal average cost function** $\bar{C}'(x)$ *approximates* how much the average cost will change if a business produces one more item. The **marginal average revenue function** $\bar{R}'(x)$ *approximates* how much the average revenue will change if a business sells one more item. The **marginal average profit function** $\bar{P}'(x)$ *approximates* how much the average profit will change if a business produces/sells one more item.

Practice 28. The price-demand and cost functions for the production of a cell phone are

$$p = 1800 - \frac{x}{1000}$$

and

$$C(x) = 450000 + 90x,$$

where x is the number of cell phones that can be sold at a price of p dollars per unit and $C(x)$ is the total cost (in dollars) of producing x units.

(a) Determine the revenue function $R(x)$. *Hint:* $R(x) = p \cdot x$.

(b) Determine the profit function $P(x)$.

(c) Determine the marginal cost function $C'(x)$.

(d) Determine the marginal revenue function $R'(x)$.

(e) Determine the marginal profit function $P'(x)$.

(f) Determine the average cost function $\bar{C}(x)$.

(g) Determine the average revenue function $\bar{R}(x)$.

(h) Determine the average profit function $\bar{P}(x)$.

(i) Determine the marginal average cost function $\bar{C}'(x)$.

(j) Determine the marginal average revenue function $\bar{R}'(x)$.

(k) Determine the marginal average profit function $\bar{P}'(x)$.

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Introductory Worksheet #8: Extreme Value Theorem

This worksheet introduces the *Extreme Value Theorem*.

Motivating Questions

1. How can we use the derivative to determine the minimum and maximum values of a function?

We are often interested in determining the minimum or maximum value of a function over a given interval (e.g. minimum cost or maximum profit). To this end, we begin with the Extreme Value Theorem.

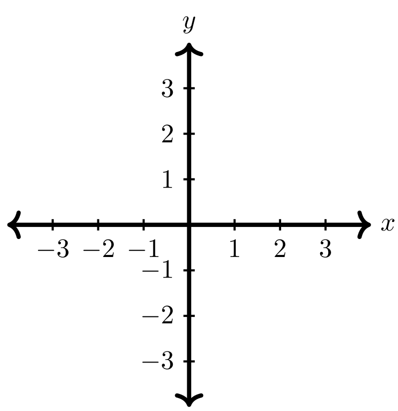
Extreme Value Theorem

If a function $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ attains a minimum and maximum value on $[a, b]$.

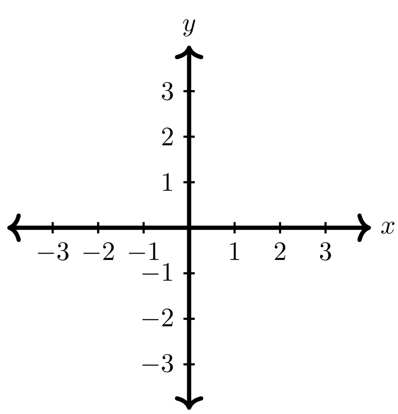
Notice the assumptions are that (1) the function is defined on a closed interval and (2) the function is continuous on that interval.

Practice 29.

(a) Draw a graph of a continuous function on an open interval (interval of the form (a, b)) that does not attain a minimum or maximum value.

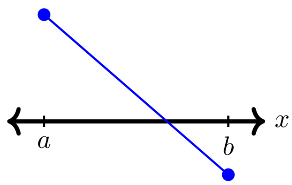


(b) Draw a graph of a discontinuous function that does not attain a minimum or maximum value.

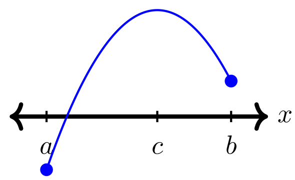


While the Extreme Value Theorem tells us that a continuous function attains a maximum and minimum value on a closed interval $[a, b]$, it does not tell us *how to find the maximum or minimum value.* Looking at the graphs below, it's not hard to see that the minimum and maximum values will occur:

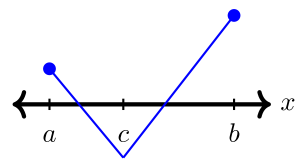
- at an endpoint a or b,
- at an interior point where $f'(x) = 0$, or
- at an interior point where $f'(x)$ does not exist.



Minimum at $x = b$.
Maximum at $x = a$.



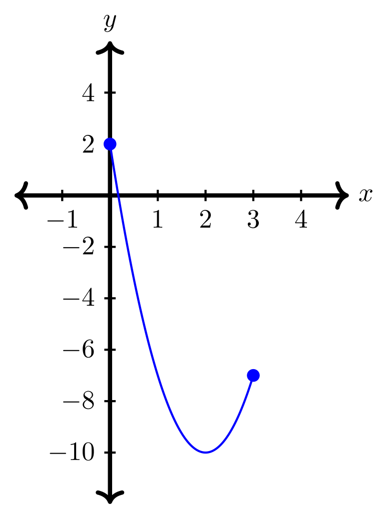
Minimum at $x = a$.
Maximum at $x = c$.



Minimum at $x = c$.
Maximum at $x = b$.

Practice 30. Determine the minimum and maximum values of $f(x) = 3x^2 - 12x + 2$ on the interval $[0, 3]$.

(a) Is $f(x)$ continuous over the interval $[0, 3]$?



(b) Determine $f'(x)$.

(c) Determine the values of x in the interval $(0, 3)$ such that $f'(x) = 0$.

(d) Determine the values of x in the interval $(0, 3)$ such that $f'(x)$ does not exist.

(e) Evaluate the function $f(x)$ at the endpoints $x = 0$ and $x = 3$, and wherever $f'(x) = 0$ or $f'(x)$ does not exist in the interval $(0, 3)$.

(f) What are the minimum and maximum values of $f(x)$ on the interval $[0, 3]$? At which x values do the minimum and maximum occur?

Practice 31. Determine the minimum and maximum values of $g(x) = x^3 - 432x + 8$ on the interval $[-6, 13]$.

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Introductory Worksheet #9: Critical Values and Increasing/Decreasing

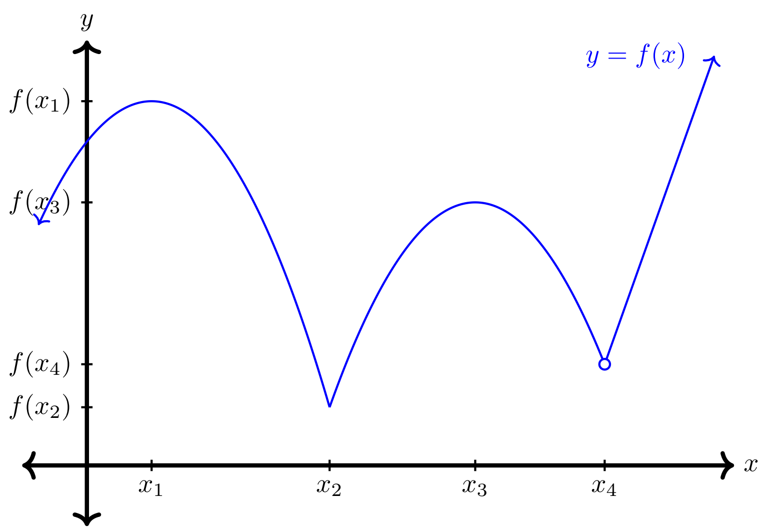
This worksheet introduces *critical values* and shows how they can be used to determine on which intervals a function is *increasing* or *decreasing*.

Motivating Questions

1. Given a function f, how do we determine its critical values?
2. Given a function f, how do we determine the intervals on which the function is increasing or decreasing?

Let $f(x)$ be a function. If $x = c$ is an interior point of the domain of f and either $f'(c) = 0$ or $f'(c)$ does not exist, then c is called a **critical value**. That is, the critical values of a function $f(x)$ are non-boundary points of the domain of f for which the derivative of f is zero or does not exist.

Practice 32. Consider the graph of a function $f(x)$ below.



(a) For which values of x is $f(x)$ undefined?

(b) For which values of x *in the domain of* f does $f'(x) = 0$? How do you know?

(c) For which values of x *in the domain of* f is $f'(x)$ undefined? How do you know?

(d) On which of the following intervals is $f'(x) > 0$? Circle all that apply. How do you know?

$(-\infty, x_1)$ (x_1, x_2) (x_2, x_3) (x_3, x_4) (x_4, ∞)

(e) For each interval that you circled in part (d), is the function f increasing or decreasing on those intervals?

(f) On which of the following intervals is $f'(x) < 0$? Circle all that apply. How do you know?

$(-\infty, x_1)$ (x_1, x_2) (x_2, x_3) (x_3, x_4) (x_4, ∞)

(g) For each interval that you circled in part (f), is the function f increasing or decreasing on those intervals?

As seen in Practice 32,

- if f is differentiable on an interval I and $f'(x) > 0$ for all $x \in I$, then f is *increasing* on I.
- if f is differentiable on an interval I and $f'(x) < 0$ for all $x \in I$, then f is *decreasing* on I.

Note that the sign of f' can change only at a point where f is undefined, or either $f'(x) = 0$ or $f'(x)$ does not exist. Then, the steps for determining the intervals on which a function $f(x)$ is increasing or decreasing are provided below.

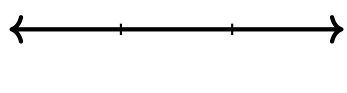
Step 1. Determine the values of x for which f is undefined and the critical values of f.

Step 2. Perform a number line test by checking the sign of f' on each interval determined by the values from Step 1.

Practice 33. Let $f(x) = 6x^3 - 18x + 3$.

(a) Determine the values of x for which $f(x)$ is undefined. Then determine the critical values of f by determining the values of x in the domain of f for which $f'(x) = 0$ or $f'(x)$ does not exist.

(b) On the number line below, label each tick mark with a value from part (a) (in ascending order). Then pick a test value within each interval and write that value on the lines provided. Finally, check the sign of $f'(x)$ on each interval by plugging the test values that you chose in for x. If the sign is positive, write a $+$ above that interval. If the sign is negative, write a $-$ above that interval.



(c) Determine the intervals over which f is increasing/decreasing by looking at the sign of f' on each interval.

f is increasing on the interval(s) ______________________________.

f is decreasing on the interval(s) ______________________________.

Practice 34. Let $g(x) = \dfrac{x-5}{x+8}$. Determine the intervals over which g is increasing/decreasing.

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Introductory Worksheet #10: Concavity and Curve Sketching

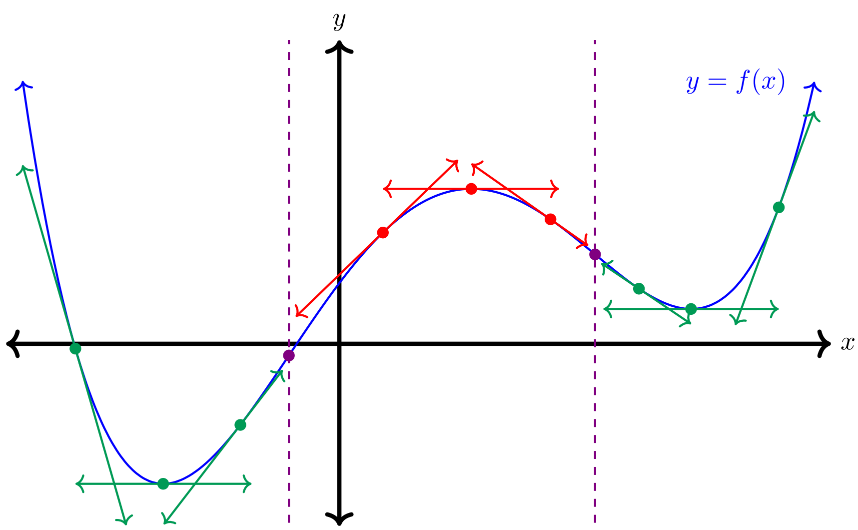
This worksheet introduces *concavity* and *inflection points.*

Motivating Questions

1. Given a function f, how do we determine the intervals on which the function is concave up or concave down?
2. Given a function f, how do we determine its inflection points?

We've seen that the first derivative of a function provides information about where the function is increasing and decreasing. What graphical information, if any, can be obtained from the second derivative? Suppose $f(x)$ is differentiable on an interval I:

- The graph of $y = f(x)$ is **concave up** on I if all the tangents to the curve on I are below the graph of $y = f(x)$.
- The graph of $y = f(x)$ is **concave down** on I if all the tangents to the curve on I are above the graph of $y = f(x)$.



Practice 35. The following questions concern the above graph.

(a) Note that the graph of $y = f(x)$ is concave up to the left of the first purple line and to the right of the second purple line. Looking at the green tangent lines in either region, are the slopes increasing or decreasing?

(b) Note that the graph of $y = f(x)$ is concave down between the two purple lines. Looking at the red tangent lines, are the slopes increasing or decreasing?

Based on the results of Practice 35, we can characterize the concavity of a function $f(x)$ by considering the rate of change of the derivative of the function ($f''(x)$). Suppose $f(x)$ is twice differentiable on an interval I:

- The graph of $y = f(x)$ is **concave up** on an interval I if $f''(x) > 0$ for all $x \in I$.
- The graph of $y = f(x)$ is **concave down** on an interval I if $f''(x) < 0$ for all $x \in I$.

A point on the graph at which the concavity changes is called an **inflection point** (the purple points on the previous graph).

Note that the sign of f'' can change only at a point where $f''(x) = 0$ or $f''(x)$ does not exist. Then, the steps for determining the intervals on which a function f is concave up or concave down are provided below.

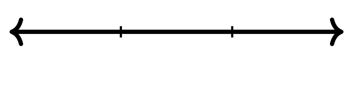
Step 1. Determine the values of x for which $f''(x) = 0$ or $f''(x)$ does not exist.

Step 2. Perform a number line test by checking the sign of f'' on each interval determined by the values from Step 1.

Practice 36. Let $f(x) = x^4 - 14x^3 + 72x^2 + 4x - 8$.

(a) Determine the values of x for which $f''(x) = 0$ or $f''(x)$ does not exist.

(b) On the number line below, label each tick mark with a value from part (a) (in ascending order). Then pick a test value within each interval and write that value on the lines provided. Finally, check the sign of $f''(x)$ on each interval by plugging the test values that you chose in for x. If the sign is positive, write a $+$ above that interval. If the sign is negative, write a $-$ above that interval.



(c) Determine the intervals over which f is concave up/concave down by looking at the sign of f'' on each interval.

f is concave up on the interval(s) ______________________________.

f is concave down on the interval(s) ______________________________.

Practice 37. Let $g(x) = \dfrac{x-5}{x+8}$. Determine the intervals over which g is concave up/concave down. [Note that this is the same function as in Practice 34.]

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Introductory Worksheet #11: Optimization

This worksheet introduces how the first derivative can be used to *optimize* a quantity given a constraint.

Motivating Questions

1. Given a constraint (or constraints) on some variables, how can we maximize or minimize a quantity related to those variables?

We'll jump right in by having your TA go through the next practice problem with you. Pay attention to the steps needed as these steps will be used in each optimization problem.

Practice 38. Suppose the product of x and y is 30 and both x and y are positive. What is the minimum possible sum of x and y?

Step 1. Identify the variables.

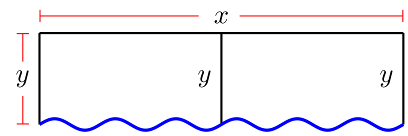
Step 2. State the *objective function*, as well as whether we are trying to maximize or minimize this quantity.

Step 3. Determine any constraints on the variables and solve for one variable in terms of the other. Are there any bounds on the independent variable?

Step 4. Rewrite the objective function from Step 2 as a function of one variable, including the domain.

Step 5. Determine the critical values of the objective function from Step 4 and perform a number line test to answer the question.

Practice 39. A farmer has 1800 ft of fencing and wants to construct a rectangular pen with a partition as shown in the diagram below. She will build the fence along a river, so she won't need fencing on that side. What is the largest area she can enclose?



Step 1. Identify the variables.

Step 2. State the *objective function*, as well as whether we are trying to maximize or minimize this quantity.

Step 3. Determine any constraints on the variables and solve for one variable in terms of the other. Are there any bounds on the independent variable?

Step 4. Rewrite the objective function from Step 2 as a function of one variable, including the domain.

Step 5. Determine the critical values of the objective function from Step 4 and perform a number line test to answer the question.

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Introductory Worksheet #12: The Idea of the Integral

This worksheet introduces how the area bounded by a curve and the x-axis relates to *definite integrals*.

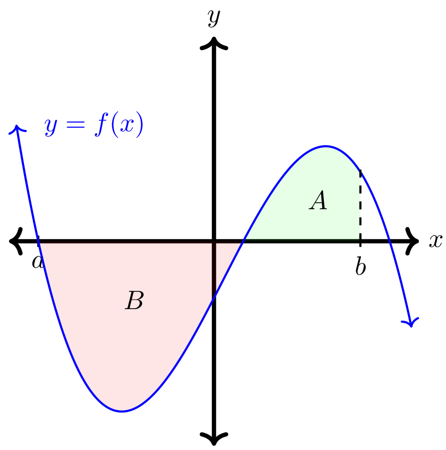
Motivating Questions

1. Given the graph of a function, how can we evaluate a definite integral?

The **definite integral**

$$\int_a^b f(x)\,dx \quad (a \le b)$$

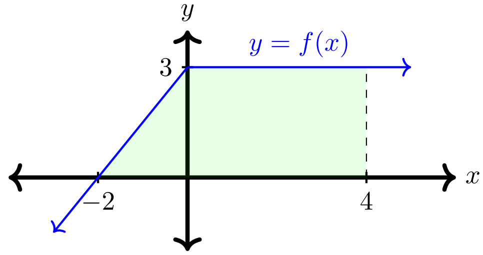
is the *signed area* between the graph of $y = f(x)$ and the x-axis from $x = a$ to $x = b$. The way to read this notation is as follows: "The definite integral from $x = a$ to $x = b$ ($\int_a^b$) of the function $f(x)$ with respect to x (dx)". When we say signed area, we mean that any area above the x-axis is considered a positive area and any area below the x-axis is considered a negative area.



$$\int_a^b f(x)\,dx = A - B$$

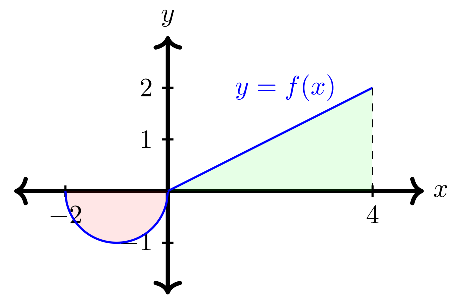
Practice 40. Use the graph to evaluate $\displaystyle\int_{-2}^{4} f(x)\,dx$.

(a) Evaluate the definite integral by separating the green area into two pieces, a triangle and a rectangle.

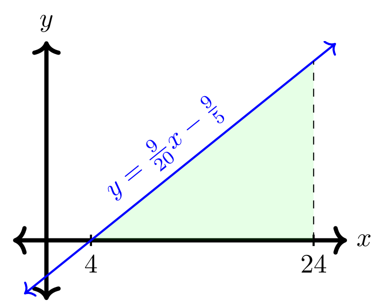


(b) Evaluate the definite integral by using the formula for the area of a trapezoid.

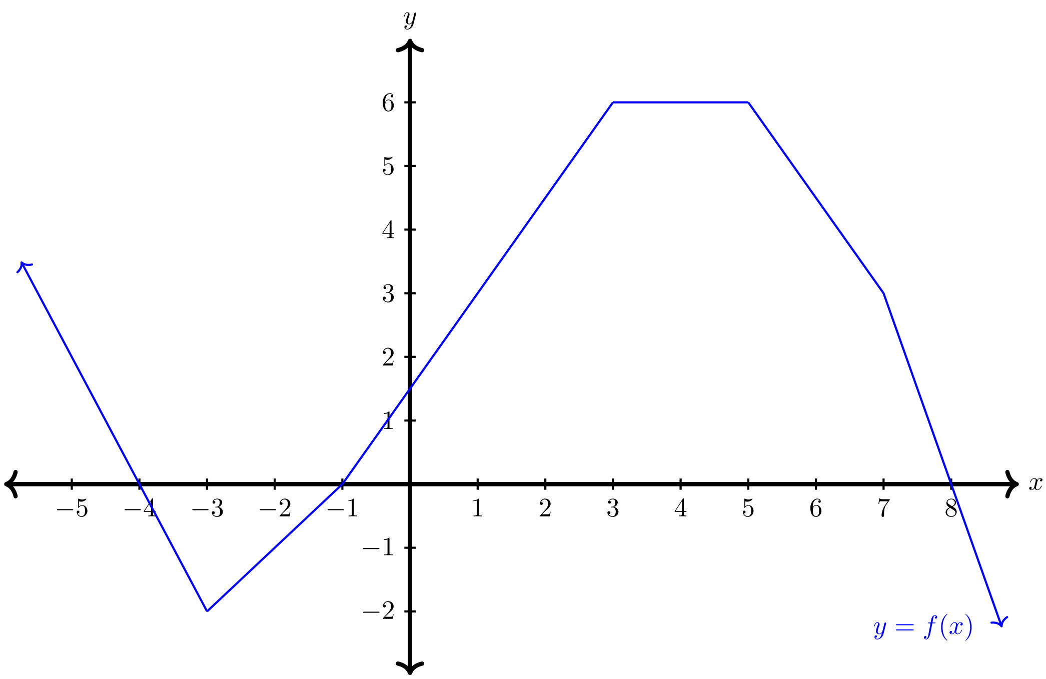
Practice 41. Use the graph to evaluate $\int_{-2}^{4} f(x)\,dx$.



Practice 42. Determine the signed area between the graph of $y = \frac{9}{20}x - \frac{9}{5}$ and the x-axis from $x = 4$ to $x = 24$.



Practice 43. Compute each definite integral using the graph of $y = f(x)$.



(a) $\int_3^5 f(x)\,dx$

(b) $\int_{-4}^1 f(x)\,dx$

(c) $\int_1^7 f(x)\,dx$

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Introductory Worksheet #13: Formulas for Antiderivatives and u-Substitution

This worksheet introduces formulas for determining *antiderivatives*.

Motivating Questions

1. Given a function f, how do we determine its antiderivative?

We have seen how to evaluate definite integrals using areas of familiar geometric shapes. In each example that we've seen, the region bounded by the graph of $y = f(x)$ and the x-axis has been either a rectangle, triangle, trapezoid, or semicircle. We will now consider a more general method, using antiderivatives, that can handle more complicated regions and does not require the graph of $f(x)$.

Rather than graphing $y = f(x)$, definite integrals can be computed by relating them to antiderivatives (provided a closed form exists). Let $F(x)$ be a function such that $F'(x) = f(x)$. Then the **antiderivative** of $f(x)$, denoted $\int f(x)\,dx$, is

$$\int f(x)\,dx = F(x) + C,$$

where C is a constant. An integral without bounds is referred to as an indefinite integral, where the output is a function (recall that the output of a definite integral is a number). The way to think about antiderivatives is that given a function f, you want to determine the most general function whose derivative is f.

Practice 44. Determine the derivative of the following functions.

(a) $F_1(x) = x^3$

(b) $F_2(x) = x^3 + 10$

(c) $F_3(x) = x^3 - 10$

(d) $F_4(x) = x^3 + \pi$

(e) $F_5(x) = x^3 + C$, where C is any real number

Based on the above results, the antiderivative of $f(x) = 3x^2$ is

$$\int 3x^2\,dx = \boxed{x^3 + C}.$$

Just like we had some rules for determining derivatives of certain types of functions, below are a few rules for determining antiderivatives of certain types of functions.

The Power Rule ($n \neq -1$)

Let $n \neq -1$. The antiderivative of $f(x) = x^n$ is $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$. That is, given a function $f(x) = x^n$ with $n \neq -1$, the antiderivative is obtained by first increasing the exponent by 1 and then dividing by the new exponent. For example, $\int x^3\,dx = \dfrac{x^4}{4} + C$.

Practice 45. Determine the following antiderivatives using the power rule.

(a) $\int (x^4 - 3x^2 + 2x + 5)\,dx$

(b) $\int x^2(x^4 + 3)\,dx$

Hint: Start by distributing x^2 to both x^4 and 3, then apply the power rule to the resulting terms.

$n = -1$ case

The antiderivative of $f(x) = x^{-1} = \dfrac{1}{x}$ is $\int \dfrac{1}{x}\,dx = \ln|x| + C$.

Practice 46. Determine $\int -9x^{-1}\,dx$ using the $n = -1$ case.

The Exponential Rule

The antiderivative of $f(x) = e^x$ is $\int e^x\,dx = e^x + C$.

Practice 47. Determine $\int 4e^x\,dx$ using the exponential rule.

Sometimes we will have to use a tool called **u-substitution** to determine an antiderivative. Recall from the chain rule for derivatives that the derivative of a composite function $f(g(x))$ is $g'(x) \cdot f'(g(x))$. If we are able to recognize the function inside an integral in the form $g'(x) \cdot f'(g(x))$, we can reverse the chain rule by letting

$$u = g(x) \quad \text{and} \quad du = g'(x)\,dx.$$

Making these substitutions,

$$\int g'(x) \cdot f'(g(x))\,dx = \int f'(u)\,du = f(u) + C.$$

The key here is to notice that the integrand involves a composite function and letting the new variable u be the "inner" function. For example, consider $\int 2xe^{x^2}\,dx$. Notice that $2x$ is the derivative of x^2, which is the "inner" function in the composite function e^{x^2}. If we let $u = x^2$ and $du = 2x\,dx$, then

$$\int 2xe^{x^2}\,dx = \int e^{u}\,du = e^u + C.$$

Hence, $\int 2xe^{x^2}\,dx = e^u + C = e^{x^2} + C$. The next problem highlights the steps for using u-substitution to determine an antiderivative.

Practice 48. Determine $\int 24x^3\sqrt{6x^4+5}\,dx$.

(a) The composite function is $\sqrt{6x^4+5}$, with "inner" function $6x^4+5$. Let $\boxed{u = 6x^4+5}$. Then what is $\dfrac{du}{dx}$?

$$\frac{du}{dx} = \underline{\qquad\qquad}$$

(b) Take the result from part (a) and multiply both sides by dx.

$$\boxed{du = \underline{\qquad\qquad}}$$

(c) Use the two boxed results (u and du) to rewrite the integral in terms of u. [Your final answer should not have any x's in it.]

$$\int \boxed{24x^3}\sqrt{\boxed{6x^4+5}}\,\boxed{dx} = \int \underline{\qquad\qquad}$$

(d) Determine the antiderivative of the result from part (c).

(e) Recall that $u = 6x^4+5$. Rewrite the result from part (d) by replacing u with $6x^4+5$. This is the antiderivative of $f(x) = 24x^3\sqrt{6x^4+5}$.

(f) Double check the result from part (e) by taking its derivative.

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Introductory Worksheet #14: Fundamental Theorem of Calculus (Part I)

This worksheet introduces the first part of the *Fundamental Theorem of Calculus.*

Motivating Questions

1. What is the Fundamental Theorem of Calculus (Part I), and how do we use it?

We have now reached the culminating theorem of the course, which we will discuss in two parts. The first part shows two things:

1. Differentiation is the inverse operation of integration.
2. Every continuous function has an antiderivative.

The Fundamental Theorem of Calculus (Part I): Let $f(t)$ be a continuous function on $[a, b]$ and let

$$F(x) = \int_a^x f(t)\,dt,$$

where $x \in (a, b)$. Then

$$F'(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x).$$

Let's make two observations:

1. $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$ shows that if we first integrate $f(t)$ from $t = a$ to $t = x$, and then differentiate with respect to x, the derivative undoes the integral. Hence, differentiation is the inverse operation of integration.
2. $F'(x) = f(x)$ shows that F is an antiderivative of f.

Practice 49. Let $F(x) = \int_0^x \left(t^3 + 6t^2 + 3t + 4\right)\,dt$. Determine $F'(x)$ using the Fundamental Theorem of Calculus (Part I).

Practice 50. Let $G(x) = \int_1^x \sqrt{t^2 - 5t + 6.25}\, dt$. For which value of x does $G'(x) = 0$?

Practice 51. For which value of x does $H(x) = \int_{-40}^{x} \big(|t| + 200\big)\, dt$ take its maximum value on the interval $[-40, 70]$?

The first part of the Fundamental Theorem of Calculus can be generalized as follows:

$$\frac{d}{dx}\left(\int_a^{g(x)} f(t)\, dt\right) = g'(x) \cdot f(g(x)).$$

This result is obtained using the chain rule with the Fundamental Theorem of Calculus (Part I):

$$\text{Inner: } g(x) \quad \xrightarrow{d/dx} \quad \boxed{g'(x)}$$

$$\text{Outer: } \int_a^x f(t)\, dt \quad \xrightarrow{d/dx} \quad f(x) \quad \xrightarrow{\text{inner}} \quad \boxed{f(g(x))}$$

Multiplying the two boxed results yields $g'(x) \cdot f(g(x))$.

Practice 52. Let $P(x) = \int_0^{5x^3+4} \left(t^2 + t + 1\right)\, dt$. Determine $P'(x)$ using the generalized version of the Fundamental Theorem of Calculus (Part I).

Date: ____________________

MA 123

Introductory Worksheet #15: Fundamental Theorem of Calculus (Part II)

This worksheet introduces the second part of the *Fundamental Theorem of Calculus*.

Motivating Questions

1. What is the Fundamental Theorem of Calculus (Part II), and how do we use it?

The second part of the Fundamental Theorem of Calculus shows two things:

1. A definite integral of a continuous function can be evaluated by evaluating any antiderivative of the function at the endpoints of the interval of integration.
2. Integration is (in some sense) the inverse operation of differentiation.

The Fundamental Theorem of Calculus (Part II): Let $f(x)$ be a continuous function on $[a, b]$ and $F(x)$ be *any* antiderivative of $f(x)$ on $[a, b]$, so that

$$F'(x) = f(x)$$

for all $x \in [a, b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b.$$

The notation $F(x)\Big|_a^b$ means that we should evaluate the function $F(x)$ at $x = b$ (the upper limit), and subtract the value of the function $F(x)$ evaluated at $x = a$ (the lower limit).

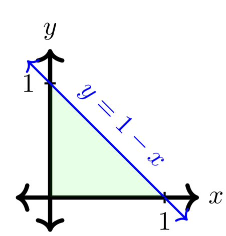
Let's make two observations:

1. $\int_a^b f(x)\,dx = F(b) - F(a)$ shows that the value of the definite integral is the difference between any antiderivative of f evaluated at the endpoints of the interval of integration (upper bound minus lower bound). *That is, we can evaluate a definite integral without graphing the given function*
2. $\int_a^x \frac{d}{dt}\left[F(t)\right] dt = F(x) - F(a)$ shows that if we first differentiate $F(t)$ with respect to t, and then integrate from $t = a$ to $t = x$, the integral undoes the derivative up to a constant $F(a)$. Hence, integration is the inverse operation of differentiation up to a constant.

The next practice problem illustrates how to use the Fundamental Theorem of Calculus (Part II) to evaluate a definite integral.

Example 1. Let $f(x) = 1 - x$.

(a) Use the graph of $y = f(x)$ to evaluate $\int_0^1 (1 - x)\,dx$.



(b) Evaluate $\int_0^1 (1-x)\,dx$ by determining an antiderivative $F(x)$ for $f(x) = 1 - x$ and evaluating $F(1) - F(0)$. Compare this answer with your answer from part (a).

Practice 53. Evaluate $\int_4^T \frac{19}{\sqrt{x}}\,dx$ using the Fundamental Theorem of Calculus (Part II).

Practice 54. Evaluate $\displaystyle\int_0^x \left(t^3 + 6t^2 + 3t + 4\right)\,dt$ using the Fundamental Theorem of Calculus (Part II). Use the answer to determine $\displaystyle\frac{d}{dx}\left(\int_0^x (t^3 + 6t^2 + 3t + 4)\,dt\right)$, and compare your result with Practice 49.

Practice 55. Evaluate $\displaystyle\int_0^8 \sqrt{t+1}\,dt$ using the Fundamental Theorem of Calculus (Part II). *Hint: Use a u-substitution. Think carefully about the endpoints of the interval of integration when you switch from t to u.*