MA 137 — Calculus 1 with Life Science Applications
Linear Approximations
(Section 4.8)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

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Tangent Line Approximation

Assume that $y = f(x)$ is differentiable at $x = a$; then

$$L(x) = f(a) + f'(a)(x - a)$$

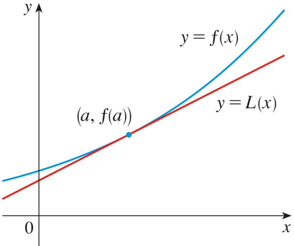
is the tangent line approximation, or **linearization**, of f at $x = a$.

Geometrically, the linearization of f at $x = a$ is the equation of the tangent line to the graph of $f(x)$ at the point $(a, f(a))$.

If $|x - a|$ is sufficiently small, then $f(x)$ can be linearly approximated by $L(x)$; that is,

$$f(x) \approx f(a) + f'(a)(x - a).$$

This approximation is illustrated in the picture on the right:



Example 1: (Nuehauser, Example # 1, p. 194)

(a) Find the linear approximation of $f(x) = \sqrt{x}$ at $x = a$.

(b) use your answer in (a) to find an approximate value of $\sqrt{26}$.

Example 2: (Online Homework HW16, # 18)

Find the linearization $L(x)$ of the function $g(x) = x\,f(x^2)$ at $x = 2$ given the following information:

$$f(2) = 1 \qquad f'(2) = 10 \qquad f(4) = 5 \qquad f'(4) = -2$$

Example 3: (Nuehauser, Problem # 34, p. 199)

Plant Biomass: Suppose that a certain plant is grown along a gradient ranging from nitrogen-poor to nitrogen-rich soil.

Experimental data show that the average mass per plant grown in a soil with a total nitrogen content of 1000 mg nitrogen per kg of soil is 2.7 g and the rate of change of the average mass per plant at this nitrogen level is 1.05×10^{-3} g per mg change in total nitrogen per kg soil.

Use a linear approximation to predict the average mass per plant grown in a soil with a total nitrogen content of 1100 mg nitrogen per kg of soil.

Example 4: (Nuehauser, Example # 3, p. 195)

Suppose $N = N(t)$ represents a population size at time t and the rate of growth as a function of N is $g(N)$.

Find the linear approximation of the growth rate at $N = 0$.

[**Hint:** We can assume that $g(0) = 0$. Indeed, when the population has size $N = 0$, its grow rate will be zero.]

[**Remark:** Your answer should show that for small population sizes, the population grows approximately exponentially.]

Example 5: (Neuhauser, Problem # 33, p. 199)

Plant Biomass: Suppose that the specific growth rate of a plant is 1%; that is, if $B(t)$ denotes the biomass at time t, then

$$\frac{1}{B(t)}\frac{dB}{dt} = 0.01$$

Suppose that the biomass at time $t = 1$ is equal to 5 grams.

Use a linear approximation to compute the biomass at time $t = 1.1$.

Higher Order Approximations

The tangent linear approximation $L(x) = f(a) + f'(a)(x - a)$ is the best first-degree (linear) approximation to $f(x)$ near $x = a$ because $f(x)$ and $L(x)$ have the same value and the same rate of change at a

$$L(a) = f(a) \qquad L'(a) = f'(a).$$

For a better approximation than a linear one, let's try to find better approximations by looking for an nth-degree polynomial

$$T_n = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots + c_n(x - a)^n$$

such that T_n and its first n derivatives have the same value at $x = a$ as f and its first n derivatives at $x = a$.
We can show that the resulting polynomial is

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

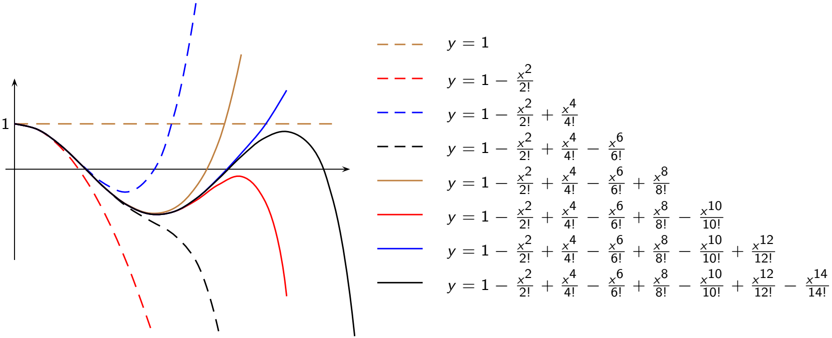
It is called the n**th-degree Taylor polynomial** of f centered at $x = a$.

Approximation of $\cos x$ centered at $a = 0$

Consider the graph of the polynomial

$$T_{2n}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^{n-1}\frac{x^{2(n-1)}}{(2n-2)!} + (-1)^n\frac{x^{2n}}{(2n)!}.$$

As n increases, the graph of $T_{2n}(x)$ appears to approach the one of $\cos x$. This suggests that we can approximate $\cos x$ with $T_{2n}(x)$ as $n \to \infty$.

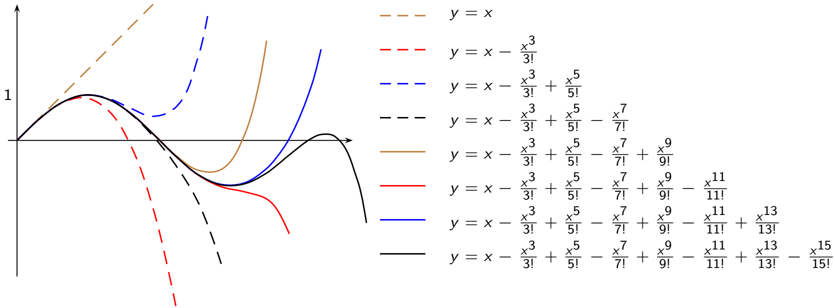


Approximation of $\sin x$ centered at $a = 0$

Consider the graph of the polynomial

$$T_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

As n increases, the graph of $T_{2n+1}(x)$ appears to approach the one of $\sin x$. This suggests that we can approximate $\sin x$ with $T_{2n+1}(x)$ as $n \to \infty$.

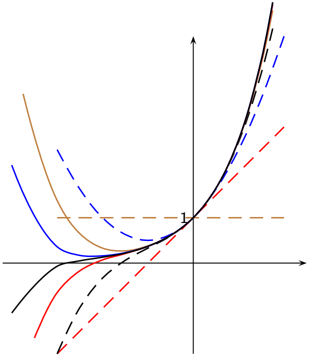


Approximation of e^x centered at $a = 0$

Consider the graph of the polynomial

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}.$$

As n increases, the graph of $T_n(x)$ appears to approach the one of e^x.
This suggests that we can approximate e^x with $T_n(x)$ as $n \to \infty$.



brown dashed	$y = 1$
red dashed	$y = 1 + x$
blue dashed	$y = 1 + x + \frac{x^2}{2!}$
black dashed	$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$
brown solid	$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$
red solid	$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$
blue solid	$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}$
black solid	$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}$

Example 6: (Bloomberg Business, 10/23/15)

Google parent Alphabet Inc. reached a record share price a day after reporting better-than-projected quarterly revenue and profit fueled by increased ad sales and a tighter lid on costs. [...] The actual figure that the company announced for the share buyback was unusually specific: \$5,099,019,513.59. Turns out, those numbers correspond to the square root of 26, or the number of letters in the English alphabet.

— ... —

Let $f(x) = \sqrt{x}$ and $a = 25$. The 5th-degree Taylor polynomial of f centered at 25 can be shown to be

$$T_5(x) = 5 + \frac{1}{10}(x-25) - \frac{1}{1,000}(x-25)^2 + \frac{1}{50,000}(x-25)^3 - \frac{1}{2,000,000}(x-25)^4 + \frac{1}{71,428,571.43}(x-25)^5$$

We can then check that

$$\sqrt{26} \approx T_5(26) = 5 + \frac{1}{10} - \frac{1}{1,000} + \frac{1}{50,000} - \frac{1}{2,000,000} + \frac{1}{71,428,571.43} = 5.099019514$$

This means that we overestimated Alphabet Inc. buyback by 41¢.