

11. (Zombie Apocalypse)

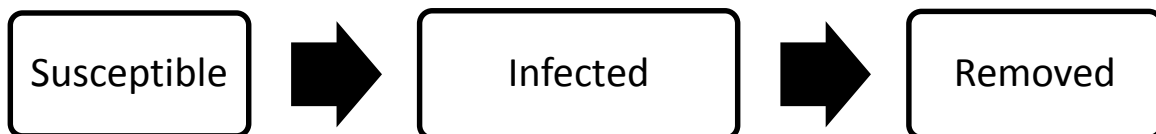
Human epidemics are often spread by contact with infectious people. There are many kinds of contagious diseases, such as H1N1 flu, smallpox, polio, measles, and rubella, which are easily spread through casual contact. Other diseases, such as Ebola, require more intimate contact. An important difference between some of these diseases is that they confer immunity to someone who recovers from it and some not. In other words, once you recover from rubella, you cannot catch it again. However, you can catch the flu many times. In this section we will formulate our first model of the spread of a disease to which people are susceptible, then infectious, and finally, recovered and immune.

These diseases divide the population into three separate groups: those that are susceptible to the disease because they have not had it, those that are infected, and those that are recovered and are subsequently immune to the disease. We denote by S the number of susceptible individuals, by I the number of infected individuals, and by R the number of recovered individuals. Here we will explore a mathematical model (called the SIR model) that attempts to capture this structure, and you will use it to study an epidemic – the zombie plague.

Basic Assumptions: We will make the following assumptions in formulating our model:

1. *SIR: All individuals fit into one of the following categories:*
 - Susceptible: those who can catch the disease
 - Infectious: those who can spread the disease
 - Removed: those who are immune and cannot spread the disease
2. *The population is large but fixed in size and confined to a well-defined region.* You might imagine the population to be a large university during the semester, when relatively little outside travel takes place.
3. *The population is well mixed;* ideally, everyone comes in contact with the same fraction of people in each category every day. Again, imagine the multitude of contacts a student makes daily at a large university.

The categories of people and the way we have assumed that they move between the categories can be summarized in the graphic “compartments”.



Members of the susceptible population acquire the disease and move into the infected population. After a period of time, those that are sick get well and move into the

recovered population. Those that are removed, stay removed.

Variables:

t = the time in days with $t=0$ at the start of observation

S = the number of susceptible individuals

I = the number of infectious individuals

R = the number of removed individuals

Time t is the independent variable; and S , I , and R are dependent on time. In other words, they are functions of time, but they are not functions given by explicit formulas. Even though we will be able to compute $S = S(t)$, there is no explicit formula for S in terms of t .

Recovery/removal: A person with rubella is infectious for about 11 days. The infectious compartment will contain people who have had the disease for 1 day, 2 days, and so on up to 11 days. At 12 days an individual moves from the infectious to the removed compartment. We will assume that the different groups in the infectious compartment are roughly the same size so that we can make a simple statement about recovery: For rubella, the number of people added to the removed compartment tomorrow is 1/11th of the infectious compartment today,

$$\text{new people in } R\text{-compartment tomorrow } \Delta R = \frac{1}{11} I$$

A general disease, from which one recovers and gains immunity, has the day's change in the recovered compartment given by, new people in R -compartment tomorrow is bI where the parameter b controls how quickly people recover, or $1/b$ is the number of days one stays in the infectious compartment. For rubella, $b = 1/11$.

Infection: If you are infectious with rubella and you sit through a whole class next to a person who is susceptible, then you will probably spread the disease to that person. If you sit through class next to an immune person, you will not spread the disease. We are supposing that our population is well mixed, so people contact others from each compartment every day. The disease-causing contacts by a single infectious person is given by the number of "close contacts" made with susceptible people during the period of infectiousness. We will assume that we can find an average number of such contacts for everyone and denote this number by a parameter c

c = average number of contacts per infected person during infectious period
Some diseases are more contagious than others, so we want a parameter that describes the communicability. The parameter c is a combined measure of the level of mixing in the population and the communicability of the disease. The problem is determining a value for c ?

The number of contacts per infective each day is given by $a = c \cdot b$, because b equals the

reciprocal of the number of days a person is infectious. This means that if I people are infectious on a particular day, then $a \cdot I$ will be the total number of adequate contacts per day. Only the contacts with susceptible people result in a new case of the disease, so we multiply this total times the fraction of susceptible people. Our final parameter, n , is the fixed total population. This makes the number of new cases of the disease,

$$\text{new cases tomorrow} = a \frac{S}{N} I.$$

Parameters for the S-I-R model:

- b = one over the average number of days being infectious
- c = average total number of contacts per infectious person
- $a = b \cdot c$ is the daily rate of contacts per infectious person
- N = total number of people in the population

The units of a , b , and c are:

- c = average total number of contacts per infectious person
- b = one over days or $1/b$ = number of days being infectious
- $a = \frac{\text{number (of adequate contacts)}}{\text{number (infectives)} \times \text{days (infectious)}}$

New cases cause the size of S to decrease and the size of I to increase by the same amount. Recoveries cause the size of I to decrease and the size of R to increase by that amount. It is best to write all our equations in terms of increases in the variables S , I , R .



Change in states:

$$\text{increase in } S: \quad \Delta S = -a \frac{S}{N} I$$

$$\text{increase in } I: \quad \Delta I = a \frac{S}{N} I - bI$$

$$\text{increase in } R: \quad \Delta R = bI$$

Exercise 1: Use the daily changes above to compute 4 days of a rubella epidemic. You need the specific parameters, a , b , and c .

- (a) For rubella, we know that $1/b = 11$ and the average contact number is $c = 6.8$. Use these values to compute the parameter a .
- (b) Suppose a population initially (at $t = 0$) has

$$S = 15,000$$

$$I = 1,500$$

$$R = 20,000$$

Use the equations of change above to complete a table showing the first four

days of the epidemic.

t	0	1	2	3	4
S	10,000				
I	1,000				
R	19,000				

(c) Use your completed table to graph S versus t , I versus t , and R versus t all on the same graph.

Discrete $S-I-R$ model

If S_0, I_0, R_0 denote the number of susceptible, infected and recovered at time 0, then we will let S_1, I_1, R_1 denote the numbers of susceptible, infected and recovered one unit of time later. So S_n, I_n, R_n denote the numbers of susceptible, infected and recovered after n units of time. Our change equations then give us that

$$S_{n+1} = S_n - a \frac{S_n}{N} I_n$$

$$I_{n+1} = a \frac{S_n}{N} I_n - b I_n$$

$$R_{n+1} = b I_n$$

Zombie Plague

t = the time in days with $t=0$ at the start of observation

S = the number of susceptible, or uninfected humans

Z = the number of zombies – the walking dead

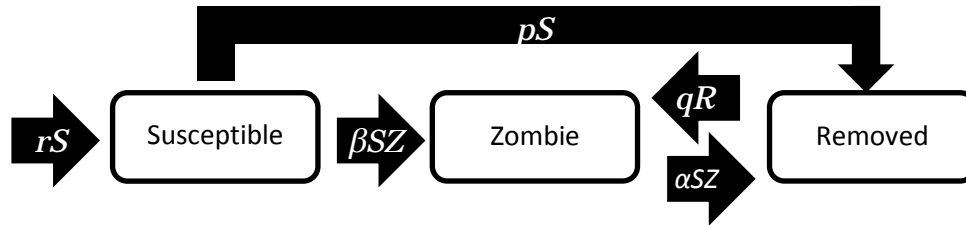
R = the number of removed – the dead

Susceptibles can become deceased through ‘natural’ causes, i.e., non-zombie-related death (parameter p). The removed class consists of individuals who have died, either through attack or natural causes. Humans in the removed class can resurrect and become a zombie (parameter q). Susceptibles can become zombies through transmission via an encounter with a zombie (transmission parameter b). Only humans can become infected through contact with zombies, and zombies only have a craving for human flesh so we do not consider any other life forms in the model. New zombies can only come from two sources:

- The resurrected from the newly deceased (removed group).
- Susceptibles who have ‘lost’ an encounter with a zombie.

We assume the birth rate is a constant, r . Zombies move to the removed class upon being ‘defeated’. This can be done by removing the head or destroying the brain of the zombie (parameter a). We also assume that zombies do not attack/defeat other zombies. Thus, the basic model is given by

$$\begin{aligned}\Delta S &= rS - \beta SZ - pS \\ \Delta Z &= \beta SZ + qR - \alpha SZ \\ \Delta R &= pS + \alpha SZ - qR\end{aligned}$$



We need estimates for α , β , p , q and r .

Birth rate, r : We should assume that the birth rate will be reduced so set $r = 0.0001$, for a birth rate of 0.01%.

Non zombie death rate, p : We will assume that $p = 0.0001$

Resurrection rate, q : Assume that this is rare and put $q = 0.0001$

We have our S-Z-R equations:

$$\begin{aligned}S_{n+1} &= rS_n - \beta S_n Z_n - pS_n \\ Z_{n+1} &= \beta S_n Z_n + qR_n - \alpha S_n Z_n \\ R_{n+1} &= pS_n + \alpha S_n Z_n - qR_n\end{aligned}$$

Exercise 2: You are going to want to use MS Excel® or its equivalent to help compute these numbers and plot the graphs. Use the values of $r = p = q = 0.0001$, above and start with 500000 humans and 1 zombie: $S_0 = 500000$, $Z_0 = 1$, $R_0 = 0$. Put your value of r in B1, p in B2, q in B3, S_0 in B4, and Z_0 in B5. This will allow you to try different scenarios more easily.

- For the first trial run, let $\alpha = 0.005$ and $\beta = 0.0095$. Put these values in B6 and B7.
- Put the values of S_0 in D1, Z_0 in E1 and R_0 in F1. Do this by putting the cursor in D1 and typing “=B4” (without the quotes) for S_0 ; in E1 type “=B5” for Z_0 ; and in F1 type “0” for R_0 .
- We need to find S_1 from the equation $S_{n+1} = rS_n - \beta S_n Z_n - pS_n$. Put your cursor in D2 and type: “=B1* D1- B7* D1* E1-B2*D1”.
- We need to do the same for Z_1 and R_1 .
- For Z_1 we use $Z_{n+1} = \beta S_n Z_n + qR_n - \alpha S_n Z_n$ and in E2 we type: “=B7* D1*E1+B3* F1- B6*D1*E1”.
- For R_1 we use $R_{n+1} = pS_n + \alpha S_n Z_n - qR_n$ and in F2 we type: “=B2* D1+B6*D1*E1- B3* F1”.
- Highlight D2, E2, F2 and copy. Paste these down the D column for 20 time units – down to D20.

Projects for MA 137

- (h) What happens to the humans (S), the zombies (Z), and the dead (R)? Draw a graph with your data in Excel and explain what you see.
- (i) Modify your data by putting let $\alpha = 0.0095$ and $\beta = 0.005$. Make another graph and explain what you see.
- (j) What happens if the death rate doubles, $p = 0.0002$?
- (k) What happens if more dead are resurrected to be zombies, $q = 0.001$?