# MA 137 – Calculus 1 with Life Science Applications

## **Discrete-Time Models**

## Sequences and Difference Equations: **Limits**

## (Section 2.2)

Department of Mathematics
University of Kentucky

# Long-Term Behavior

When studying populations over time, we are often interested in their long-term behavior.

Specifically, if $N_t$ is the population size at time $t$, $t = 0, 1, 2, \ldots$, we want to know how $N_t$ behaves as $t$ increases, or, more precisely, as $t$ tends to infinity.

Using our general setup and notation, we want to know the behavior of $a_n$ as $n$ tends to infinity and use the shorthand notation

$$\lim_{n \longrightarrow \infty} a_n$$

which we read as 'the limit of $a_n$ as $n$ tends to infinity.'

# Definition and Notation

## Definition (**Informal)**

We say that the limit as $n$ tends to infinity of a sequence $a_n$ is a number $L$, written as $\lim_{n \longrightarrow \infty} a_n = L$, if we can make the terms $a_n$ as close to $L$ as we like by taking $n$ sufficiently large.

## Definition (**Formal)**

The sequence $\{a_n\}$ has a limit $L$, written as $\lim_{n \longrightarrow \infty} a_n = L$, if, for any given any number $d > 0$, there is an integer $N$ so that

$$|a_n - L| < d$$

whenever $n > N$.

If the limit exists, the sequence **converges** (or is **convergent**).

Otherwise we say that the sequence **diverges** (or is **divergent**).

# Example 1:

Let $a_n = \frac{1}{n}$ for $n = 1, 2, 3, \ldots.$

Show that $\lim_{n \longrightarrow \infty} \frac{1}{n} = 0$

# Example 2:

Let $a_n = (-1)^n$ for $n = 0, 1, 2, \ldots$.

Show that $\lim_{n \longrightarrow \infty} (-1)^n$ does not exist.

What about the limit of the sequence $b_n = \cos(\pi n)$ ?

# Limit Laws

The operations of arithmetic, namely, addition, subtraction, multiplication, and division, all behave reasonably with respect to the idea of getting closer to as long as nothing illegal happens.

This is summarized by the following laws:

If $\lim_{n \longrightarrow \infty} a_n$ and $\lim_{n \longrightarrow \infty} b_n$ exist and $c$ is a constant, then

1. $\lim_{n \longrightarrow \infty} (a_n + b_n) = (\lim_{n \longrightarrow \infty} a_n) + (\lim_{n \longrightarrow \infty} b_n)$
2. $\lim_{n \longrightarrow \infty} (c\, a_n) = c\,(\lim_{n \longrightarrow \infty} a_n)$
3. $\lim_{n \longrightarrow \infty} (a_n b_n) = (\lim_{n \longrightarrow \infty} a_n)(\lim_{n \longrightarrow \infty} b_n)$
4. $\lim_{n \longrightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \longrightarrow \infty} a_n}{\lim_{n \longrightarrow \infty} b_n}$, provided $\lim_{n \longrightarrow \infty} b_n \neq 0$

# Example 3:

Find $\lim_{n \longrightarrow \infty} \frac{n(1-3n^2)}{n^3+1}$.

Find $\lim_{n \longrightarrow \infty} \frac{n}{n^2+1}$.

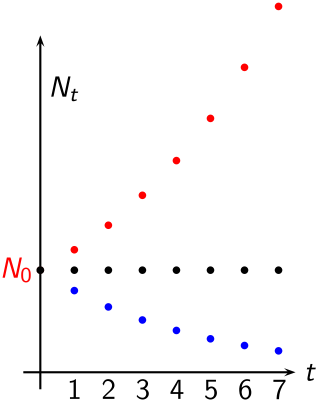
# Example 4:

For $R > 0$, we know that exponential growth is given by

$$N_t = N_0 R^n \qquad n = 0, 1, 2, \ldots$$

The figure below indicates that

$$\lim_{n \longrightarrow \infty} N_t = \begin{cases} 0 & \text{if } 0 < R < 1 \\ N_0 & \text{if } R = 1 \\ \infty & \text{if } R > 1 \end{cases}$$



# Example 5:

Find $$\lim_{n \longrightarrow \infty} \frac{3 \cdot 4^n + 1}{4^n}$$

# Squeeze (Sandwich) Theorem for Sequences

Sometimes the limit of a sequence can be difficult to calculate and we need to employ some other techniques. One of those techniques is to use the Squeeze (Sandwich) Theorem for Sequences.

## Squeeze (Sandwich) Theorem for Sequences

Consider three sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ and suppose there exists an integer $N$ such that

$$a_n \leq b_n \leq c_n \qquad \text{for all} \quad n > N.$$

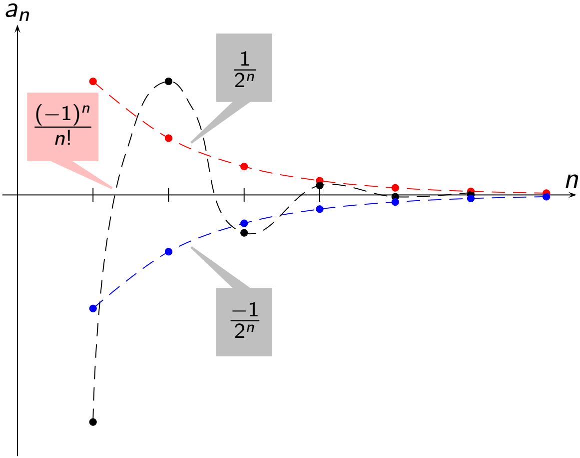
If $\lim_{n \longrightarrow \infty} a_n = L = \lim_{n \longrightarrow \infty} c_n$ then $\lim_{n \longrightarrow \infty} b_n = L.$

The values in the following table and the graph on the left

| $n$ | $1/n!$ | $1/2^n$ |
|---|---|---|
| 1 | 1 | 0.5 |
| 2 | 0.5 | 0.25 |
| 3 | $0.1\overline{6}$ | 0.125 |
| 4 | $0.041\overline{6}$ | 0.0625 |
| 5 | $0.008\overline{3}$ | 0.03125 |
| 6 | $0.0013\overline{8}$ | 0.015625 |
| 7 | 0.000198 | 0.0078125 |
| ⋮ | ⋮ | ⋮ |

suggest that for $n \geq 4$ we have

$$\frac{-1}{2^n} \leq \frac{(-1)^n}{n!} \leq \frac{1}{2^n} \qquad n \geq 4.$$




So by the Squeeze Theorem it follows that

$$\lim_{n \longrightarrow \infty} (-1)^n \frac{1}{n!} = 0.$$

# Example 6:

Find $\lim_{n \longrightarrow \infty} \frac{2n + (-1)^n}{n}$

# Example 7:

Find $\displaystyle\lim_{n\longrightarrow\infty} \frac{5^n}{n!}$