MA 137 – Calculus 1 with Life Science Applications
The Product and Quotient Rule
and the Derivatives of Rational and Power Functions
(Section 4.4)

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Basic Rules (cont'd)

Theorem

Suppose $f(x)$ and $g(x)$ are differentiable functions. Then the following relationships hold:

4. $$\frac{d}{dx}[f(x)\cdot g(x)] = \frac{d}{dx}[f(x)]\cdot g(x) + f(x)\cdot\frac{d}{dx}[g(x)]$$

(*in prime notation*) $(fg)'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$

5. $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)]\cdot g(x) - f(x)\cdot\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

(*in prime notation*) $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{[g(x)]^2}$

The Power Rule for Negative Exponents

The quotient rule allows us to extend the power rule to the case where the exponent is a negative integer:

Theorem

If $f(x) = x^{-n}$, where n is a positive integer, then $f'(x) = -nx^{-n-1}$.

Proof: We write $f(x) = \dfrac{1}{x^n}$ and use the quotient rule

$$f'(x) = \frac{0 \cdot x^n - 1 \cdot nx^{n-1}}{[x^n]^2} = -\frac{nx^{n-1}}{x^{2n}} = -nx^{(n-1)-2n} = -nx^{-n-1}.$$

There is a general form of the power rule in which the exponent can be any real number. In Section 4.4, we give the proof for the case when the exponent is rational; we prove the general case in Section 4.7.

Theorem (General Form)

If $f(x) = x^r$, where r is any real number, then $f'(x) = r\,x^{r-1}$.

Example 1: (Neuhauser, Example # 1, p. 161)

Differentiate $f(x) = (3x + 1)(2x^2 - 5)$.

Example 2: (Online Homework HW12, # 17)

Differentiate $Y(u) = (u^{-2} + u^{-3})(u^5 + u^2)$.

Example 3: (Neuhauser, Problem # 39, p. 166)

Assume that $f(x)$ is differentiable.

Find an expression for the derivative of

$$y = -5x^3 f(x) - 2x$$

at $x = 1$, assuming that $f(1) = 2$ and $f'(1) = -1$.

Example 4: (Online Homework HW12, # 19)

Differentiate $$f(x) = \frac{ax+b}{cx+d} \quad ,$$

where a, b, c, and d are constants and $ad - bc \neq 0$.

Example 5: (Online Homework HW12, # 22)

Find an equation of the tangent line to the given curve at the specified point:

$$y = \frac{\sqrt{x}}{x+3} \qquad P(4, 2/7).$$

Example 6: (Neuhauser, Example # 6, p. 163)

Differentiate the Monod growth function

$$f(R) = \frac{aR}{k + R}$$

where a and k are positive constants.

Example 7: (Neuhauser, Problem # 84, p. 167)

Assume that $f(x)$ is differentiable.

Find an expression for the derivative of

$$y = \frac{f(x)}{x^2 + 1}$$

at $x = 2$, assuming that $f(2) = -1$ and $f'(2) = 1$.

Proofs:

4. We use the definition of the derivative, rewrite the numerator in a 'tricky' way and use the limit laws and the continuity of the functions.

$$
\begin{aligned}
(fg)'(x) &\overset{\text{def}}{=} \lim_{h\to 0} \frac{(fg)(x+h) - (fg)(x)}{h} \\
&\overset{\text{def}}{=} \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&\overset{\text{trick}}{=} \lim_{h\to 0} \frac{f(x+h)g(x+h) \boxed{-f(x)g(x+h) + f(x)g(x+h)} - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h}\, g(x+h) + f(x)\, \frac{g(x+h) - g(x)}{h} \right] \\
&\overset{\text{rule}}{=} \left[\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \right] \left[\lim_{h\to 0} g(x+h) \right] + f(x) \left[\lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \right] \\
&\overset{\text{cont.}}{=} f'(x)\, g(x) + f(x)\, g'(x).
\end{aligned}
$$

5. We use the definition of the derivative, rewrite the numerator in a 'tricky' way and use the limit laws and the continuity of the functions.

$$(f/g)'(x) =$$

$$\overset{\text{def}}{=} \lim_{h\to 0} \frac{(f/g)(x+h) - (f/g)(x)}{h}$$

$$\overset{\text{def}}{=} \lim_{h\to 0} \frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h} = \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{h\,g(x)\,g(x+h)}$$

$$\overset{\text{trick}}{=} \lim_{h\to 0} \frac{f(x+h)g(x) \boxed{-f(x)g(x) + f(x)g(x)} - f(x)g(x+h)}{h\,g(x)\,g(x+h)}$$

$$= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h\,g(x+h)} - f(x)\,\frac{g(x+h) - g(x)}{h\,g(x)g(x+h)} \right]$$

$$\overset{\text{rule}}{=} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h\to 0} \frac{1}{g(x+h)} - \lim_{h\to 0} \frac{f(x)}{g(x)g(x+h)} \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$\overset{\text{cont.}}{=} f'(x)\frac{1}{g(x)} - \frac{f(x)}{[g(x)]^2}g'(x) = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}.$$