MA 137 – Calculus 1 with Life Science Applications
Implicit Differentiation
(Section 4.6)

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Implicit Differentiation

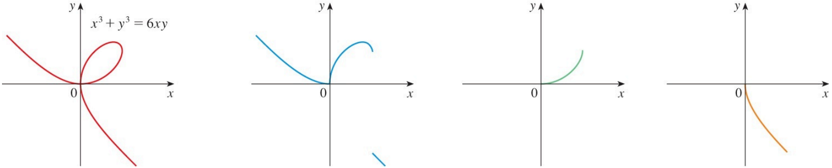
So far, we have considered only functions of the form $y = f(x)$, which define y explicitly as a function of x.

It is also possible to define y implicitly as a function of x, as in the following equation:

$$x^3 + y^3 = 6xy \tag{1}$$

Here, y is still given as a function of x (i.e., y is the dependent variable), but there is no obvious way to solve for y.

Below are the graphs of three such functions related to equation (1), dubbed the folium of Descartes.



When we say that f is implicitly defined by the equation given in (1), we mean that the equation

$$x^3 + [f(x)]^3 = 6\,x\,f(x)$$

is true for all values of x in the domain of f.

Fortunately, there is a very useful technique, based on the chain rule, that will allow us to find dy/dx for implicitly defined functions.

This technique is called **implicit differentiation**.

We summarize the steps we take to find dy/dx when an equation defines y implicitly as a differentiable function of x:

1. Differentiate both sides of the equation with respect to x, keeping in mind that y is a function of x.

   [**Note:** differentiating terms involving y typically requires the chain rule.]

2. Solve the resulting equation for dy/dx.

Example 1:

(a) Find y' if y is implicitly defined by $x^3 + y^3 = 6xy$.

(b) Find an equation for the tangent line to the folium of Descartes $x^3 + y^3 = 6xy$ at the point $(3, 3)$.

Example 2: (Online Homework HW14, # 2)

Given $xy + 2x + 3x^2 = -4$:

(a) Find y' by implicit differentiation.

(b) Solve the equation for y and differentiate to get y' in terms of x.
(The answers should be consistent!)

Example 3: (Neuhauser, Problem # 8, p. 179)

Find dy/dx by implicit differentiation if

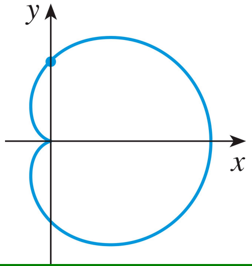
$$\frac{x}{xy+1} = 2xy.$$

Example 4: (Online Homework HW14, # 6)

Use implicit differentiation to find an equation of the tangent line to the curve (called **cardioid**)

$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2$$

at the point $(0, 1/2)$.



Power Rule for Rational Exponents

We now provide a proof of the generalized form of the power rule when the exponent r is a rational number: $\frac{d}{dx}(x^r) = r\,x^{r-1}$.

We write $r = p/q$, where p and q are integers and are in lowest terms. (If q is even, we require x and y to be positive.) Then

$$y = x^r \qquad \Longleftrightarrow \qquad y = x^{p/q} \qquad \Longleftrightarrow \qquad y^q = x^p.$$

Differentiating both sides of $y^q = x^p$ with respect to x, we find that

$$qy^{q-1}\frac{dy}{dx} = px^{p-1}.$$

Hence

$$\begin{aligned} \frac{dy}{dx} &= \frac{p}{q}\,\frac{x^{p-1}}{y^{q-1}} = \frac{p}{q}\,\frac{x^{p-1}}{(x^{p/q})^{q-1}} = \frac{p}{q}\,\frac{x^{p-1}}{x^{p-p/q}} = \frac{p}{q}\,x^{p-1-(p-p/q)} \\ &= \frac{p}{q}\,x^{p/q-1} = r\,x^{r-1} \end{aligned}$$