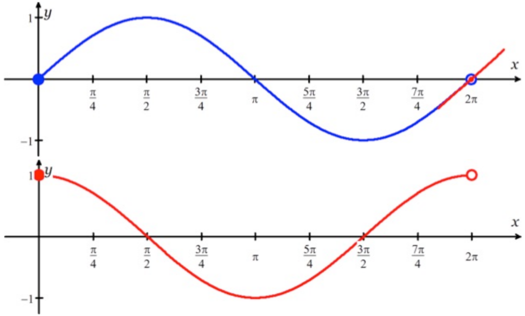
MA 137 — Calculus 1 with Life Science Applications
Derivatives of Trigonometric Functions
(Section 4.8)

Department of Mathematics
University of Kentucky

Cyclic phenomena are most easily modeled by sines and cosines:

- length of day;
- length of season;
- some population models (e.g. ideal predator-prey models).

We need to know how fast they change.
Let's compare $\sin x$ and $\cos x$:



The Derivative of Sine and Cosine

Theorem

The functions $\sin x$ *and* $\cos x$ *are differentiable for all* x*, and*

$$\frac{d}{dx}\sin x = \cos x \qquad \textit{and} \qquad \frac{d}{dx}\cos x = -\sin x$$

We need the trigonometric limits from Section 3.4 to compute the derivatives of the sine and cosine functions. Namely,

$$\lim_{h\to 0}\frac{\sin h}{h} = 1 \qquad \text{and} \qquad \lim_{h\to 0}\frac{\cos h - 1}{h} = 0.$$

We also need the addition formulas for sine and cosine

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Note that all angles are measured in radians.

Proof for Cosine

We use the formal definition of derivatives:

$$
\begin{aligned}
\frac{d}{dx}\cos x &\overset{\text{def}}{=} \lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h} \\
&\overset{\text{add. form.}}{=} \lim_{h\to 0} \frac{\cos x\cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x\,(\cos h - 1) - \sin x \sin h}{h} \\
&= \lim_{h\to 0} \left[\cos x\,\frac{\cos h - 1}{h} - \sin x\,\frac{\sin h}{h}\right] \\
&\overset{\text{laws}}{=} \cos x \lim_{h\to 0}\frac{\cos h - 1}{h} - \sin x \lim_{h\to 0}\frac{\sin h}{h} \\
&\overset{\text{fund. lim.}}{=} \cos x\cdot 0 - \sin x \cdot 1 \\
&= -\sin x
\end{aligned}
$$

Proof for Sine

We use the formal definition of derivatives:

$$
\begin{aligned}
\frac{d}{dx}\sin x &\overset{def}{=} \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&\overset{\text{add. form.}}{=} \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\,(\cos h-1)+\cos x\sin h}{h} \\
&= \lim_{h\to 0}\left[\sin x\,\frac{\cos h-1}{h}+\cos x\,\frac{\sin h}{h}\right] \\
&\overset{\text{laws}}{=} \sin x\lim_{h\to 0}\frac{\cos h-1}{h}+\cos x\lim_{h\to 0}\frac{\sin h}{h} \\
&\overset{\text{fund. lim.}}{=} \sin x\cdot 0+\cos x\cdot 1 \\
&= \cos x
\end{aligned}
$$

Derivatives of Remaining Trigonometric Functions

The derivatives of the other trigonometric functions can be found using the following identities and the quotient rule:

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

For example: $\frac{d}{dx}(\tan x) = \cdots = \sec^2 x = 1 + \tan^2 x.$

Example 1: (Online Homework HW15, # 3)

Find the equation of the tangent line to the curve $y = 6x\cos x$ at the point $(\pi, -6\pi)$.

Example 2: (Online Homework HW15, # 4)

(a) Let $f(x) = \sin^3(x)$. Find $f'(x)$.

(b) Let $g(x) = \sin(x^3)$. Find $g'(x)$.

Example 3: (Online Homework HW15, # 7)

Find the derivative of the following function:

$$f(x) = \frac{\cos(2x)}{6 - \sin(2x)}$$

Example 4: (Online Homework HW15, # 8)

Find the derivative of the following function:

$$f(x) = \left(x^3 - \cos(6x^2)\right)^5$$

Example 5:

Human heart goes through cycles of contraction and relaxation (called systoles). During cycles, blood pressure goes up and down repeatedly; as heart contracts, pressure rises, and as heart relaxes (for a split second), pressure drops.
Consider approximate function for blood pressure of a patient

$$P(t) = 100 + 20\cos\left(\frac{\pi t}{35}\right) \;\text{ mmHg}$$

where t is measured in minutes . Find and interpret $P'(t)$.

Example 6: (Online Homework HW15, # 9)

During the human female menstrual cycle, the gonadotropin, FSH or follicle stimulating hormone, is released from the pituitary in a sinusoidal manner with a period of approximately 28 days.
Guyton's text on Medical Physiology shows that if we define day 0 ($t = 0$) as the beginning of menstruation, then FSH, $F(t)$, cycles with a high concentration of about 4.4 (relative units) around day 9 and a low concentration of about 1.2 around day 23.

a. Consider a model of the concentration FSH (in relative units) given by

$$F(t) = A + B\cos(\omega(t - \varphi)),$$

where A, B, ω, and φ (with $0 \le \varphi \le 28$) are constants and t is in days. Use the data above to find the four parameters.

If ovulation occurs around day 14, then what is the approximate concentration of FSH at that time?

You should sketch a graph of the concentration of FSH over one period.

b. Find the derivative of $F(t)$. Give its value at the time of ovulation ($t = 14$).