MA 137 — Calculus 1 with Life Science Applications
Derivatives of Exponential Functions
(Section 4.9)

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The Derivative of the Natural Exponential Function

Theorem

The function e^x *is differentiable for all* x*, and* $\frac{d}{dx} e^x = e^x$.

In particular, if $g(x)$ *is a differentiable function, it follows from the chain rule that*

$$\frac{d}{dx} e^{g(x)} = e^{g(x)} \cdot g'(x).$$

We need to know the following limit to compute the derivative of the natural exponential function. Namely,

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

Although we cannot rigorously prove this result here, the table below should convince you of its validity

h	-0.1	-0.01	-0.001	$\cdots$	0.001	0.01	0.1
$\frac{e^h - 1}{h}$	0.9516	0.9950	0.9995		1.0005	1.0050	1.0517

Proof

We use the formal definition of the derivative. In the final step, we will be able to write the term e^x in front of the limit because e^x does not depend on h.

$$
\begin{aligned}
\frac{d}{dx} e^x &\overset{\text{def}}{=} \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\
&\overset{\text{exp. prop.}}{=} \lim_{h \to 0} \frac{e^x e^h - e^x}{h} \\
&= \lim_{h \to 0} \frac{e^x (e^h - 1)}{h} \\
&\overset{\text{laws}}{=} e^x \lim_{h \to 0} \frac{e^h - 1}{h} \\
&\overset{\text{fund. lim.}}{=} e^x \cdot 1 \\
&= e^x
\end{aligned}
$$

The Derivative of ANY Exponential Function

Theorem

The function a^x *is differentiable for all* x, *and* $\frac{d}{dx}\, a^x = a^x \cdot \ln a$.

In particular, if $g(x)$ *is a differentiable function, it follows from the chain rule that*

$$\frac{d}{dx}\, a^{g(x)} = a^{g(x)} \cdot \ln a \cdot g'(x).$$

We can prove the above result using the definition of the derivative and the limit

$$\lim_{h \to 0} \frac{a^h - 1}{h} = \ln a,$$

in the same manner that we did for the natural exponential function.

Alternatively, we can use the following identity

$$a^x = e^{\ln a^x} = e^{x \ln a}$$

and the chain rule. Namely,

$$\frac{d}{dx}\, a^x = \frac{d}{dx}\, e^{x \ln a} = e^{x \ln a} \cdot \ln a = a^x \cdot \ln a.$$

Example 1: (Nuehauser, Example # 1, p. 190)

Find the derivative of $f(x) = e^{-x^2/2}$.

Example 2:

Find the derivative with respect to x of $g(x) = xe^{-x}$.

Evaluate $g'(x)$ at $x = 1$.

Example 3: (Online Homework HW15, # 14)

The cutlassfish is a valuable resource in the marine fishing industry in China. A von Bertalanffy model is fit to data for one species of this fish giving the length of the fish, $L(t)$ (in mm), as a function of the age, a (in yr). An estimate of the length of this fish is

$$L(a) = 593 - 378e^{-0.166a}.$$

(a) Find the L-intercept.
Find an equation for the horizontal asymptote of $L(a)$.
Find the maximum possible length of this fish.

(b) Determine how long it takes for this fish to reach 90 percent of its maximum length.

(c) Differentiate $L(a)$ with respect to a.

Example 4: (Neuhauser, Example # 5, p. 191)

Exponential Growth: Show that the function $N(t) = N_0 e^{rt}$ satisfies the differential equation

$$\frac{dN}{dt} = rN(t) \qquad N(0) = N_0.$$

[N_0 is the population size at time $t = 0$ and r is called the growth rate.]

Example 5: (Neuhauser, Example # 6, p. 192)

Radioactive Decay: Show that the function $W(t) = W_0 e^{-rt}$ satisfies the differential equation

$$\frac{dW}{dt} = -rW(t) \qquad W(0) = W_0.$$

[W_0 is the amount of material at time $t = 0$ and r is called the radioactive decay rate.]

Example 6: (Neuhauser, Problem # 63, p. 193)

(a) Find the derivative of the logistic growth curve (Example 4, Section 3.3, p. 123)

$$N(t) = \frac{K}{1 + \left(\frac{K}{N_0} - 1\right) e^{-rt}}$$

(b) Show that $N(t)$ satisfies the differential equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \qquad N(0) = N_0$$

(c) Plot the per capita rate of growth $\frac{1}{N}\frac{dN}{dt}$ as a function of N, and note that it decreases with increasing population size.