MA 137 — Calculus 1 with Life Science Applications
Derivatives of Logarithmic Functions and Logarithmic Differentiation
(Section 4.10)

Department of Mathematics
University of Kentucky

The Derivative of the Natural Logarithmic Function

Theorem

The function $\ln x$ *is differentiable for all* $x > 0$, *and* $\frac{d}{dx}\ln x = \frac{1}{x}$.

In particular, if $g(x)$ *is a differentiable function, it follows from the chain rule that*

$$\frac{d}{dx}\ln g(x) = \frac{1}{g(x)}\, g'(x).$$

We can use the derivative of e^x and the relationship between the exponential and the natural logarithmic functions to find the derivative of the function $\ln x$. Namely, we start by taking the derivative with respect to x of both sides of $e^{\ln x} = x$. We obtain

$$\frac{d}{dx}e^{\ln x} = \frac{d}{dx}x \iff e^{\ln x}\frac{d}{dx}\ln x = 1 \iff \frac{d}{dx}\ln x = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

Alternative Proof

We use the formal definition of the derivative and $e^x = \lim_{u\to\infty}\left(1+\frac{x}{u}\right)^u$

$$\begin{aligned}
\frac{d}{dx}\ln x &\overset{\text{def}}{=} \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} \\
&\overset{\ln\text{ prop.}}{=} \lim_{h\to 0}\frac{1}{h}\ln\left(\frac{x+h}{x}\right) \\
&= \lim_{h\to 0}\frac{1}{x}\,\frac{x}{h}\ln\left(1+\frac{1}{x/h}\right) \qquad u = x/h \\
&\overset{\text{laws}}{=} \frac{1}{x}\lim_{u\to\infty}\ln\left(1+\frac{1}{u}\right)^u \\
&\overset{\text{cont.}}{=} \frac{1}{x}\ln\left[\lim_{u\to\infty}\left(1+\frac{1}{u}\right)^u\right] \\
&= \frac{1}{x}\ln e = \frac{1}{x}
\end{aligned}$$

The Derivative of ANY Logarithmic Function

Theorem

The function $\log_a x$ *is differentiable for* $x > 0$*, and* $\frac{d}{dx} \log_a x = \frac{1}{(\ln a)x}$.

In particular, if $g(x)$ *is a differentiable function, it follows from the chain rule that*

$$\frac{d}{dx} \log_a g(x) = \frac{1}{(\ln a)g(x)} g'(x).$$

From the base change formula for logarithms we have that

$$\log_a x = \frac{\ln x}{\ln a}$$

Thus it is enough to find the derivative of $\ln x$. Hence the formula.

Example 1: (Nuehauser, Problems # 28/34/52, p. 204)

Find $\dfrac{dy}{dx}$ when $y = \ln(1 - x^2)$.

Find $\dfrac{dy}{dx}$ when $y = \left[\ln(1 - x^2)\right]^3$.

Find $\dfrac{dy}{ds}$ when $y = \ln(\ln s)$.

Example 2: (Nuehauser, Problem # 56, p. 204)

Find $\dfrac{dy}{dx}$ when $y = \log(3x^2 - x + 2)$.

[**Note:** $\log = \log_{10}$]

Example 3: (Nuehauser, Problem # 62, p. 204)

Assume that $f(x)$ is differentiable with respect to x. Show that

$$\frac{d}{dx} \ln \left[\frac{f(x)}{x} \right] = \frac{f'(x)}{f(x)} - \frac{1}{x}$$

Logarithmic Differentiation

In 1695, Leibniz introduced logarithmic differentiation, following Johann Bernoulli's suggestion to find derivatives of functions of the form

$$y = [f(x)]^x.$$

Bernoulli generalized this method and published his results two years later.

The **basic idea** is to take logarithms on both sides and then to use implicit differentiation.

Example 4: (Neuhauser, Example # 12, p. 202)

Find $\dfrac{dy}{dx}$ when $y = x^x$.

What about $\dfrac{d}{dx}\left[(2x)^{2x}\right]$?

Example 5: (Neuhauser, Problems # 68/75/76, p. 204)

Use logarithmic differentiation to find the first derivative of the functions

$$y = (\ln x)^{3x} \qquad y = x^{\cos x} \qquad y = (\cos x)^{x}$$

Example 6: (Neuhauser, Problem # 77, p. 204)

Use logarithmic differentiation to find the first derivative of the function

$$y = \frac{e^{2x}(9x-2)^3}{\sqrt[4]{(x^2+1)(3x^3-7)}}$$

Power Rule (General Form)

Theorem

Let $f(x) = x^r$, where r is any real number. Then

$$\frac{d}{dx}x^r = r\,x^{r-1}$$

Proof: We set $y = x^r$ and use logarithmic differentiation to obtain

$$\begin{aligned}\frac{d}{dx}\left[\ln y\right] &= \frac{d}{dx}\left[\ln x^r\right]\\ \frac{1}{y}\frac{dy}{dx} &= \frac{d}{dx}\left[r\ln x\right]\\ \frac{1}{y}\frac{dy}{dx} &= r\frac{1}{x}\end{aligned}$$

Solving for dy/dx yields

$$\frac{dy}{dx} = r\frac{1}{x}y = r\frac{1}{x}x^r = r\,x^{r-1}$$