

MA 137 — Calculus 1 with Life Science Applications
Extrema and The Mean Value Theorem
(Section 5.1)

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The Mean Value Theorem (MVT)

The Mean Value Theorem is a very important in calculus. Its consequences are far reaching, and we will use it to derive important results that will help us to analyze functions.

Theorem (Mean Value Theorem)

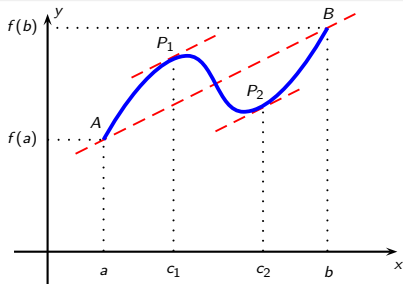
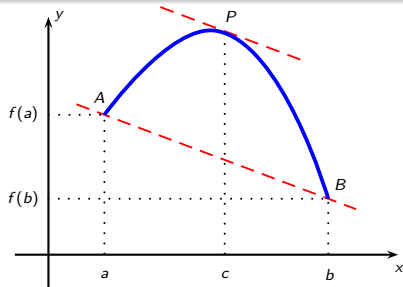
If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there exists at least one number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Geometrically, it says that there exists a point $P(c, f(c))$ on the graph where the tangent line at this point is parallel to the secant line through $A(a, f(a))$ and $B(b, f(b))$.

The MVT is an “existence” result: It tells us neither how many such points there are nor where they are in the interval (a, b) .

Geometric Interpretation and a Special Case



The proof of the MVT is typically done by first showing a special case of the theorem called Rolle's Theorem.

You can read its proof on p. 211 of the Neuhauser book.

Theorem (Rolle's Theorem – 1691)

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) , and if $f(a) = f(b)$, then there exists a number $c \in (a, b)$ such that $f'(c) = 0$.

The MVT follows from Rolle's theorem and is a "tilted" version of that theorem. The secant and tangent lines in the MVT are no longer necessarily horizontal, as in Rolle's theorem, but are "tilted"; they are still parallel, though.

Proof of the MVT: We define the following function:

$$F(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

The function F is continuous on $[a, b]$ and differentiable on (a, b) . Furthermore, $F(a) = f(a) = F(b)$. Hence, we can apply Rolle's theorem to the function $F(x)$. There exists a $c \in (a, b)$ with $F'(c) = 0$. Since

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

it follows that, for this value of c ,

$$0 = F'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

and hence

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example 1: (Online Homework HW17, # 10)

Graph the function $f(x) = x^3 - 2x$ and its secant line through the points $(-2, -4)$ and $(2, 4)$. Use the graph to estimate the x -coordinate of the points where the tangent line is parallel to the secant line.

Find the exact value of the numbers c that satisfy the conclusion of the Mean Value Theorem for the interval $[-2, 2]$.

Example 2: (Online Homework HW17, # 12)

Find all numbers c that satisfy the conclusion of Rolle's Theorem for the following function

$$f(x) = 9x\sqrt{x+2}$$

on the interval $[-2, 0]$.

Example 3: (Online Homework HW17, # 13)

Consider the function $f(x) = 3 - 3x^{2/3}$ on the interval $[-1, 1]$. Which of the three hypotheses of Rolle's Theorem fails for this function on the interval?

- (a) $f(x)$ is continuous on $[-1, 1]$.
- (b) $f(x)$ is differentiable on $(-1, 1)$.
- (c) $f(-1) = f(1)$.

Consequences of the MVT

We discuss two consequences of the MVT.

The first corollary is useful in obtaining information about a function on the basis of its derivative. The importance of the second corollary will become more apparent in Example 7 and Section 5.8.

Corollary 1

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) such that

$$m \leq f'(x) \leq M \quad \text{for all } x \in (a, b)$$

then

$$m(b - a) \leq f(b) - f(a) \leq M(b - a)$$

Corollary 2

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) , with $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.

Example 4: (Online Homework HW17, # 14)

Suppose $f(x)$ is continuous on $[3, 5]$ and

$$-5 \leq f'(x) \leq 2$$

for all x in $(3, 5)$.

Use the Mean Value Theorem to estimate $f(5) - f(3)$.

Example 5: (Neuhauser, Example # 8, p. 221)

Denote the population size at time t by $N(t)$, and assume that $N(t)$ is continuous on the interval $[0, 10]$ and differentiable on the interval $(0, 10)$ with $N(0) = 100$ and $\left| \frac{dN}{dt} \right| \leq 3$ for all $t \in (0, 10)$.

What can you say about $N(10)$?

Example 6: (Online Homework HW17, # 15)

Let $f(x) = 8 \sin(x)$.

(a) $|f'(x)| \leq \underline{\hspace{2cm}}$

(b) By the Mean Value Theorem,

$$|f(b) - f(a)| \leq \underline{\hspace{2cm}} |a - b|$$

for all a and b .

[**Remark:** This problem is also a variation of Example 9, Neuhauser, p. 212]

Example 7: (Neuhauser, Problem # 56, p. 224)

We have seen that $f(x) = f_0 e^{rx}$ satisfies the differential equation $\frac{df}{dx} = r f(x)$ with $f(0) = f_0$.

This exercise will show that $f(x)$ is in fact the only solution.

Suppose that r is a constant and f is a differentiable function with

$$\frac{df}{dx} = r f(x) \quad (1)$$

for all $x \in \mathbb{R}$, and $f(0) = f_0$. The following steps will show that $f(x) = f_0 e^{rx}$, $x \in \mathbb{R}$, is the only solution of (1).

- (a) Define the function $F(x) = f(x)e^{-rx}$, $x \in \mathbb{R}$. Use the product rule to show that $F'(x) = e^{-rx}[f'(x) - rf(x)]$.
- (b) Use (a) and (1) to show that $F'(x) = 0$ for all $x \in \mathbb{R}$.
- (c) Use Corollary 2 to show that $F(x)$ is a constant and, hence, $F(x) = F(0) = f_0$.
- (d) Show that (c) implies that $f_0 = f(x)e^{-rx}$ and therefore, $f(x) = f_0 e^{rx}$.