MA 137 — Calculus 1 with Life Science Applications
L'Hôpital's Rule
(Section 5.5)

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Heuristics

We have often encountered the situation in which we had to compute $\lim_{x\to a} \frac{f(x)}{g(x)}$ and we had that both the following limits were zero

$$\lim_{x\to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to a} g(x) = 0.$$

Using a linear approximation at $x = a$, we find that, for x close to a

$$\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a)(x-a)}{g(a) + g'(a)(x-a)}$$

Since $f(a) = g(a) = 0$ and $x \neq a$, the right-hand side is equal to

$$\frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}$$

provided that $f'(a)/g'(a)$ is defined. We therefore hope that something like

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

holds when $f(a)/g(a)$ is of the form $0/0$ and $f'(a)/g'(a)$ is defined. In fact, something like this does hold; it is called **l'Hôpital's rule**.

L'Hôpital's Rule

Theorem

Suppose that f and g are differentiable functions and that

$$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x) \qquad \text{or} \qquad \lim_{x\to a} f(x) = \infty = \lim_{x\to a} g(x)$$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the second limit exists.

L'Hôpital's rule can actually be applied to calculate limits for seven kinds of indeterminate expressions

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0\cdot\infty \qquad \infty-\infty \qquad 0^0 \qquad 1^\infty \qquad \infty^0.$$

(Note that l'Hôpital's rule works for $a = +\infty$ or $-\infty$ as well.)

Reduction to $0/0$ or ∞/∞ Form

$0 \cdot \infty$ Suppose we have to compute $\lim_{x \to a} f(x)g(x)$ where $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$. To apply l'Hôpital's rule to this kind of limit write it in one of the two forms

$$\lim_{x \to a} f(x)g(x) = \lim_{x \to a} \frac{f(x)}{1/g(x)} = \lim_{x \to a} \frac{g(x)}{1/f(x)}$$

In the first case the ratio is $0/0$, whereas in the second case the ratio is ∞/∞. Usually only one of the two expressions is easy to evaluate.

$\infty - \infty$ Suppose we have to compute $\lim_{x \to a}[f(x) - g(x)]$ where $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$. To apply l'Hôpital's rule to this kind of limit write it in one of the two forms

$$\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x)\left(1 - \frac{g(x)}{f(x)}\right) = \lim_{x \to a} g(x)\left(\frac{f(x)}{g(x)} - 1\right)$$

and hope that the limit is of the form $0 \cdot \infty$.

0^0 1^∞ ∞^0 Suppose we have to compute $\lim_{x \to a}[f(x)]^{g(x)}$, which becomes of the form 0^0, 1^∞ or ∞^0. The key to solving these limits is to write them as exponentials

$$\lim_{x \to a}[f(x)]^{g(x)} = \lim_{x \to a} \exp\left\{\ln\,[f(x)]^{g(x)}\right\} = \lim_{x \to a} \exp\left\{g(x) \cdot \ln f(x)\right\} = \exp\left[\lim_{x \to a}\left(g(x) \cdot \ln f(x)\right)\right].$$

The last step, in which we interchanged lim and exp, uses the fact that the exponential function is continuous.

Example 1: (Nuehauser, p. 253)

Evaluate $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$.

Example 2: (Nuehauser, p. 253)

Evaluate $\lim_{x \to 0} \frac{e^x - 1}{x}$.

Example 3: (Neuhauser, Example 3, p. 255)

Evaluate $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2}$.

Example 4: (Neuhauser, Problem # 25, p. 259)

Evaluate $\lim_{x\to\infty} x \cdot e^{-x}$.

What about $\lim_{x\to\infty} x^{13} \cdot e^{-x}$? (Online Homework HW20, # 5)

Example 5: (Online Homework HW20, # 3)

Evaluate $\lim_{x \to 0^+} 7\sqrt{x} \cdot \ln x$.

Example 6: (Neuhauser, Example 9, p. 257)

Evaluate $\lim_{x \to \infty} x - \sqrt{x^2 + x}$.

Example 7: (Online Homework HW20, # 4)

Evaluate $\lim_{x \to 0^+} x^x$.

Example 8: (Neuhauser, Problem 62, p. 259)

Use l'Hôpital's rule to find $\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x$ where c is a constant.

What about $\lim_{x\to\infty} 3x\big(\ln(x+3)-\ln x\big)$? (Online Homework HW20, #10)