MA 137 — Calculus 1 with Life Science Applications
The Definite Integral
(Section 6.1)

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Sigma (Σ) Notation

In approximating areas we have encountered sums with many terms. A convenient way of writing such sums uses the Greek letter Σ (which corresponds to our capital S) and is called *sigma notation*. More precisely, if $a_1, a_2, \dots, a_n$ are real numbers we denote the sum

$$a_1 + a_2 + \cdots + a_n$$

by using the notation

$$\sum_{k=1}^{n} a_k.$$

The integer k is called an *index* or *counter* and takes on the values $1, 2, \dots, n$. For example,

$$\sum_{k=1}^{6} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 1 + 4 + 9 + 16 + 25 + 36 = 91$$

whereas

$$\sum_{k=3}^{6} k^2 = 3^2 + 4^2 + 5^2 + 6^2 = 9 + 16 + 25 + 36 = 86.$$

Summation Rules

The rules and formulas given next allow us to compute fairly easily Riemann sums where the number n of subintervals is rather large. We can also get compact and manageable expressions for the sum so that we can readily investigate what happens as n approaches infinity.

[sr$_1$] $$\sum_{k=1}^{n} c = n\,c$$

[sr$_2$] $$\sum_{k=1}^{n} (c\,a_k) = c \sum_{k=1}^{n} a_k$$

[sr$_3$] $$\sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$$

Note: The summations rules are nothing but the usual rules of arithmetic rewritten in the Σ notation.
For example, **[sr$_2$]** is nothing but the distributive law of arithmetic

$$c\,a_1 + c\,a_2 + \cdots + c\,a_n = c\,(a_1 + a_2 + \cdots + a_n);$$

[sr$_3$] is nothing but the commutative law of addition

$$(a_1 \pm b_1) + \cdots + (a_n \pm b_n) = (a_1 + \cdots + a_n) \pm (b_1 + \cdots + b_n).$$

Formulas [Neuhauser, Example #3 (p. 279); Problem # 31 (p. 291)]

[sf$_1$] $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$ **[sf$_2$]** $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof: In the case of **[sf$_1$]**, let S denote the sum of the integers $1, 2, 3, \ldots, n$. Let us write this sum S twice: we first list the terms in the sum in increasing order whereas we list them in decreasing order the second time:

$$\begin{array}{ccccccccc} S & = & 1 & + & 2 & + & \cdots & + & n \\ S & = & n & + & n-1 & + & \cdots & + & 1 \end{array}$$

If we now add the terms along the vertical columns, we obtain

$$2S = \underbrace{(n+1) + (n+1) + \cdots + (n+1)}_{n \text{ times}} = n(n+1).$$

This gives our desired formula, once we divide both sides of the above equality by 2.

In the case of **[sf$_2$]**, let S denote the sum of the integers $1^2, 2^2, 3^2, \ldots, n^2$. The *trick* is to consider the sum $\sum_{k=1}^{n}[(k+1)^3 - k^3]$. On the one hand, this new sum collapses to

$$(2^3 - 1^3) + (3^3 - 2^3) + (4^3 - 3^3) + \cdots + (n^3 - (n-1)^3) + ((n+1)^3 - n^3) = (n+1)^3 - 1^3 = n^3 + 3n^2 + 3n$$

On the other hand, using our summation rules together with **[sf$_1$]** gives us

$$\sum_{k=1}^{n}[(k+1)^3 - k^3] = \sum_{k=1}^{n}[3k^2 + 3k + 1] = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = 3S + 3\,\frac{n(n+1)}{2} + n$$

Equating the right hand sides of the above identities gives us: $3S + 3\,\dfrac{n(n+1)}{2} + n = n^3 + 3n^2 + 3n.$

If we solve for S and properly factor the terms, we obtain our desired expression.

More Formulas

The next formulas can be verified in a sequential order using the same type of trick used in the proof for **[$\mathbf{sf_2}$]**. The proofs get increasingly more tedious.

[$\mathbf{sf_3}$] $$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

[$\mathbf{sf_4}$] $$\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

Example 1: (Online Homework, HW 23, # 15)

Find the numerical value of the sums below:

- $$\sum_{j=3}^{7}(4j-1)$$
- $$\sum_{i=3}^{5}(i^2-i)$$

Example 2:

Find the numerical value of the sums below:

- $$\sum_{j=3}^{n}(4j-1)$$
- $$\sum_{i=3}^{n}(i^2-i)$$

Back to the Area Problem: Partitions

The idea we have used so far is to "to partition" or subdivide the given interval $[a, b]$ into smaller subintervals on each of which the variable x, and thus the function $f(x)$, does not change much.

Definition of a Partition

A *partition* of an interval $[a, b]$ is a set of points $\{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}$, listed increasingly, on the x-axis with $a = x_0$ and $x_n = b$. That is:

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

These points subdivide the interval $[a, b]$ into n *subintervals*

$$[a, x_1],\ [x_1, x_2],\ [x_2, x_3],\ \ldots,\ [x_{n-1}, b].$$

The k-th subinterval is thus of the form $[x_{k-1}, x_k]$ and it has *length*

$$\Delta x_k = x_k - x_{k-1}.$$

Assumption

Set $\|P\| = \max\limits_{1 \le i \le n} \{\Delta x_i\}$. We assume that our partition P is such that $\|P\| \to 0$ as $n \to \infty$. In other words, we assume that the length of the longest (and, hence, of all) subinterval(s) tend(s) to zero whenever the number of subintervals in P becomes very large.

The Definite Integral

Let $f(x)$ be a function defined on an interval $[a,b]$.

- Partition the interval $[a,b]$ in n subintervals of lengths $\Delta x_1, \ldots, \Delta x_n$, respectively.
- For $k = 1, \ldots, n$ pick a representative point c_k in the corresponding k-th subinterval.

The **definite integral of f from a to b** is defined as

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\cdot\Delta x_k = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\cdot\Delta x_k$$

and it is denoted by $$\int_a^b f(x)\,dx.$$

The sum $\sum_{k=1}^{n} f(c_k)\cdot\Delta x_k$ is called a *Riemann sum* in honor of the German mathematician Bernhard Riemann (1826-1866), who developed the above ideas in full generality. The symbol $\int$ is called the *integral sign*. It is an elongated capital S, of the kind used in the 1600s and 1700s. The letter S stands for the summation performed in computing a Riemann sum. The numbers a and b are called the *lower and upper limits of integration*, respectively. The function $f(x)$ is called the *integrand* and the symbol dx is called the *differential* of x. You can think of the dx as representing what happens to the term Δx in the limit, as the size Δx of the subintervals gets closer and closer to 0.

- The role of x in a definite integral is the one of a *dummy variable*. In fact $\int_a^b x^2\,dx$ and $\int_a^b t^2\,dt$ have the same meaning. They represent the same number.
- We recall that a limit does not necessarily exist. However:

Theorem

If f is continuous on $[a,b]$ then $\int_a^b f(x)\,dx$ *exists.*

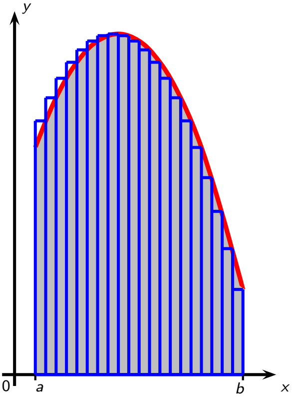
- As we observed earlier, it is computationally easier to partition the interval $[a,b]$ into n subintervals of equal length. Therefore each subinterval has length $\Delta x = \dfrac{b-a}{n}$ (we drop the index k as it is no longer necessary). In this case, there is a simple formula for the points of the partition, namely:

$$x_0 = a+0\cdot\Delta x = a,\;\; x_1 = a+\Delta x,\; \ldots\; x_k = a+k\cdot\Delta x,\; \ldots,\; x_n = a+n\cdot\Delta x = b$$

or, more concisely, $x_k = a + k\cdot\dfrac{b-a}{n}$ for $k = 0,1,2,\ldots,n.$

Definite Integrals and Areas

We stress the fact that if the function f takes on only positive values then the definite integral is nothing but the area of the region below the graph of f, lying above the x-axis, and bounded by the vertical lines $x = a$ and $x = b$.

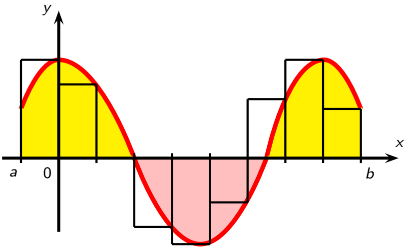


Distance traveled by an object:

If the positive valued function under consideration is the velocity $v(t)$ of an object at time t, then the area underneath the graph of the velocity function and lying above the t-axis represents the total distance traveled by the object from $t = a$ to $t = b$.

What if the Function Takes on Negative Values?

If f happens to take on both positive and negative values, then the Riemann sum is the sum of the areas of rectangles that lie above the x-axis and the negatives of the areas of rectangles that lie below the x-axis. Passing to the limit, we obtain that, in general, a definite integral can be interpreted as a difference of areas:



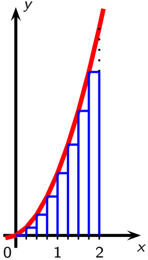
$$\int_a^b f(x)\,dx = [\text{area of the region(s) lying above the } x\text{-axis}] - [\text{area of the region(s) lying below the } x\text{-axis}]$$

Right Versus Left Endpoint Estimates

Observe that x_k, the right endpoint of the k-th subinterval, is also the left endpoint of the $(k+1)$-th subinterval. Thus the Riemann sum estimate for the definite integral of a function f defined over an interval $[a, b]$ can be written in either of the following two forms

$$\sum_{k=0}^{n-1} f(x_k) \cdot \Delta x_{k+1} \qquad \sum_{k=1}^{n} f(x_k) \cdot \Delta x_k$$

depending on whether we use left or right endpoints, respectively.

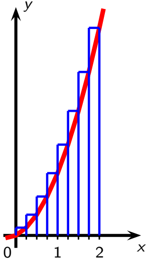


Left endpoint
Riemann sum estimate

If we are dealing with a regular partition, the above sums become

$$\sum_{k=0}^{n-1} f(a + k \cdot \Delta x) \cdot \Delta x \qquad \sum_{k=1}^{n} f(a + k \cdot \Delta x) \cdot \Delta x$$

with $\Delta x = \dfrac{b-a}{n}$ and $x_k = a + k \cdot \Delta x$ for $k = 0, 1, \ldots, n$.



Right endpoint
Riemann sum estimate

Example 3: (Online Homework, HW 23, # 11)

Express the limit as a definite integral

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(5+\frac{2i}{n}\right)^{10}$$

Example 4: (Online Homework, HW 23, # 12)

Express the limit as a definite integral

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\sqrt{1+\frac{4i}{n}}$$

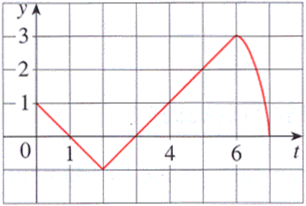
Example 5: (Online Homework, HW 23, # 7)

Evaluate the following integral by interpreting it in terms of areas:

$$\int_0^3 \left(\frac{1}{2}x - 1\right) dx$$

Example 6: (Online Homework, HW 23, # 10)

Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown below.



- Evaluate $g(x)$ for $x = 0, 1, 2, 3, 4, 5,$ and 6.
- Estimate $g(7)$.
- At what value of x does g attain its maximum?
- At what value of x does g attain its minimum?