MA 137 — Calculus 1 with Life Science Applications
The Fundamental Theorem of Calculus
(Section 6.2)

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An Example

- The easiest procedure for computing definite integrals is not by computing a limit of a Riemann sum, but by relating integrals to (anti)derivatives.
- This relationship is so important in Calculus that the theorem that describes it is called

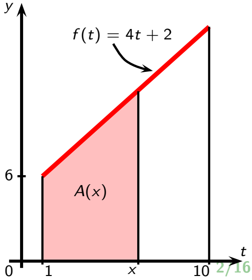
  the Fundamental Theorem of Calculus.
- We introduce the theorem by first analyzing a simple example.

Example 1

Find a formula for $A(x) = \int_1^x (4t+2)\,dt$, where $1 \le x \le 10$.

Find the values $A(5), A(10)$, and $A(1)$.

What is the derivative of $A(x)$ with respect to x?



The Main Idea of the FTC

Suppose that for *any* function $f(t)$ it were true that the area function

$$A(x) = \int_a^x f(t)\,dt$$

satisfies

$$A(a) = \int_a^a f(t)\,dt = 0 \qquad\qquad A'(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt \right) = f(x).$$

Moreover, suppose that $B(x)$ is any function such that: $B'(x) = f(x) = A'(x)$.
By a consequence of the MVT we know that there is a constant value c such that $B(x) = A(x) + c$.

All these facts put together help us easily evaluate $\displaystyle\int_a^b f(t)\,dt$.

Indeed

$$\begin{aligned}
\underset{\sim\sim\sim\sim}{\int_a^b f(t)\,dt} &= A(b) = A(b) - 0 \\
&= \underset{\sim\sim\sim\sim}{A(b) - A(a)} = [A(b)+c] - [A(a)+c] \\
&= \underset{\sim\sim\sim\sim}{B(b) - B(a)}
\end{aligned}$$

The FTC

The previous 'speculations' are actually true for any continuous function on the interval over which we are integrating. These results are stated in the following theorem, which is divided into two parts:

Theorem (The Fundamental Theorem of Calculus)

PART I: *Let $f(t)$ be a continuous function on the interval $[a,b]$. Then the function $A(x)$, defined by the formula*

$$A(x) = \int_a^x f(t)\,dt$$

for all x in the interval $[a,b]$, is an antiderivative of $f(x)$, that is

$$A'(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt \right) = f(x)$$

for all x in the interval $[a,b]$.

PART II: *Let $F(x)$ be any antiderivative of $f(x)$ on $[a,b]$, so that $F'(x) = f(x)$ for all x in the interval $[a,b]$. Then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Special Notation for Part II:

Part II of the FTC tells us that evaluating a definite integral is a two-step process:

- find *any* antiderivative $F(x)$ of the function $f(x)$; and then
- compute the difference $F(b) - F(a)$.

A notation has been devised to separate the two steps of this process: $F(x)\Big|_a^b$ stands for the difference $F(b) - F(a)$. Thus

$$\int_a^b f(x)\,dx \;=\; F(x)\Big|_a^b \;=\; F(b) - F(a).$$

About the Proof of the FTC:

We already gave an explanation of why the second part of the Fundamental Theorem of Calculus follows from the first one.
To prove the first part we need to use the definition of the derivative.

Proof of Part I

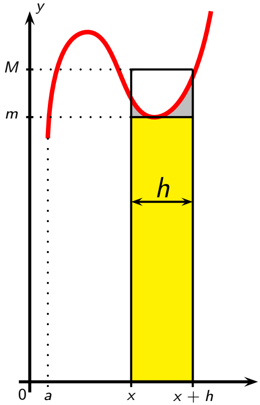
We must show that

$$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = f(x).$$

For convenience, let us assume that f is a positive valued function. Given that $A(x)$ is defined by $\displaystyle\int_a^x f(t)\,dt$, the numerator of the above difference quotient is

$$A(x+h) - A(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt.$$

Using properties **4.** and **5.** of definite integrals, the above difference equals $\displaystyle\int_x^{x+h} f(t)\,dt$.



As the function f is continuous over the interval $[x, x+h]$, the Extreme Value Theorem says that there are values c_1 and c_2 in $[x, x+h]$ where f attains the minimum and maximum values, say m and M, respectively.

Thus $\quad m \le f(t) \le M \quad$ on $[x, x+h]$. As the length of the interval $[x, x+h]$ is h, by property **6.** of definite integrals we have that

$$f(c_1)h = mh \le \int_x^{x+h} f(t)\,dt \le Mh = f(c_2)h$$

or, equivalently,

$$f(c_1) \le \frac{\displaystyle\int_x^{x+h} f(t)\,dt}{h} \le f(c_2).$$

Finally, as f is continuous we have that $\lim\limits_{h\to 0} f(c_1) = f(x) = \lim\limits_{h\to 0} f(c_2)$.

This concludes the proof.

Example 2: (Online Homework HW24, # 2)

Suppose

$$f(x) = \int_0^x \frac{t^2 - 16}{2 + \cos^2(t)}\, dt$$

For what value(s) of x does $f(x)$ have a local maximum?

Example 3: (Online Homework HW24, # 6)

Find a function f and a number a such that

$$2 + \int_a^x \frac{f(t)}{t^7}\, dt = 4x^{-3}$$

Leibniz's Rule

Combining the chain rule and the FTC (Part I), we can differentiate integrals with respect to x when the upper and/or lower limits of integration are function of x.

We summarize these facts into the following result:

Leibniz's Rule

If $g(x)$ and $h(x)$ are differentiable functions and $f(u)$ is continuous for u between $g(x)$ and $h(x)$, then

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(u)\,du = f[h(x)]h'(x) - f[g(x)]g'(x)$$

Example 4:

Compute

$$\frac{d}{dx} \int_{\sin x}^{1} u^2 \, du$$

Example 5: (Online Homework HW24, # 5)

Find the derivative of the following function

$$F(x) = \int_{x^4}^{x^6} (2t-1)^3 \, dt$$

using the Fundamental Theorem of Calculus.

Example 6: (Online Homework HW24, # 7)

Evaluate the definite integral

$$\int_4^7 \left(\frac{d}{dt} \sqrt{3 + 3t^4} \right) dt$$

using the Fundamental Theorem of Calculus.

Example 7: (Online Homework HW24, # 12)

Evaluate the definite integral

$$\int_1^4 \frac{x^2 + 5}{x}\, dx$$

Example 8: (Online Homework HW24, # 14)

Evaluate the definite integral

$$\int_0^1 (x^2 + 8 - 2e^{-2x})\, dx$$

Example 9: (Online Homework HW24, # 15)

Find the area bounded by the function $y = 1 - x^2$ and the x-axis.