MA137 – Calculus 1 with Life Science Applications
Preliminaries and Elementary Functions
(Sections 1.2 & 1.3)

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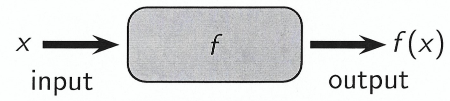
Definition of Function

A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

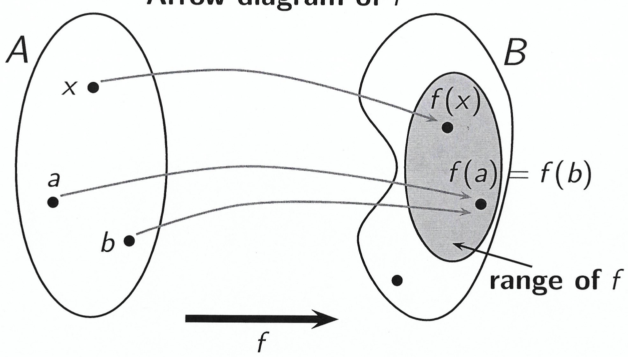
The set A is called the **domain** of f whereas the set B is called the **codomain** of f; $f(x)$ is called the **value of** f **at** x, or the **image of** x **under** f.

The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain: range of $f = \{f(x) \,|\, x \in A\}$.



Machine diagram of f

Arrow diagram of f



Notation: To define a function, we often use the notation

$$f : A \longrightarrow B, \qquad x \mapsto f(x)$$

where A and B are subsets of the set of real numbers $\mathbb{R}$.

The Domain of a Function

The domain of a function is the set of all inputs for the function.

The domain may be stated explicitly.

For example, if we write

$$f(x) = 1 - x^2 \qquad -2 \le x \le 5$$

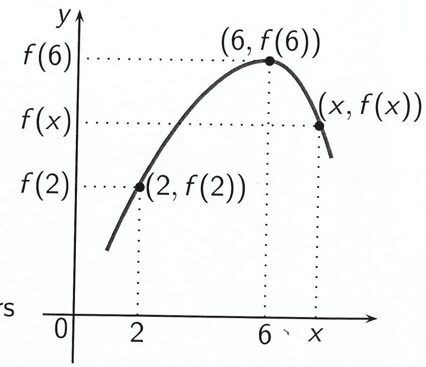
then the domain is the set of all real numbers x for which $-2 \le x \le 5$.

If the function is given by an algebraic expression and the domain is not stated explicitly, then by convention *the domain is the set of all real numbers for which the expression is defined.*

Fact: Two functions f and g are equal if and only if
1. f and g are defined on the same domain,
2. $f(x) = g(x)$ for all x in the domain.

Graphs of Functions

The graph of a function is the most important way to visualize a function. It gives a picture of the behavior or 'life history' of the function.
We can read the value of $f(x)$ from the graph as being the height of the graph above the point x.



If f is a function with domain A, then the graph of f is the set of ordered pairs

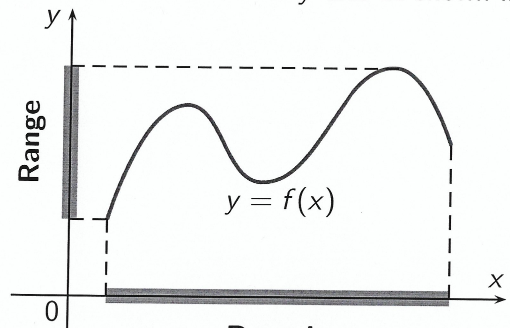
$$\textbf{graph of } f = \{(x, f(x)) \mid x \in A\}.$$

In other words, the graph of f is the set of all points (x, y) such that $y = f(x)$; that is, the graph of f is the graph of the equation $y = f(x)$.

Obtaining Information from the Graph of a Function

The values of a function are represented by the height of its graph above the x-axis. So, we can read off the values of a function from its graph.

In addition, the graph of a function helps us picture the domain and range of the function on the x-axis and y-axis as shown in the picture:

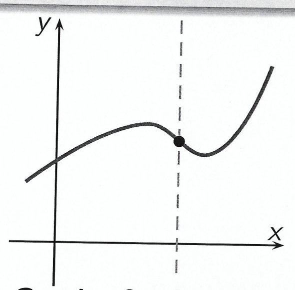


The Vertical Line Test

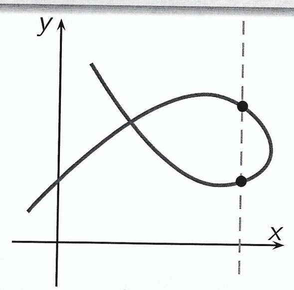
The graph of a function is a curve in the xy-plane. But the question arises: Which curves in the xy-plane are graphs of functions?

The Vertical Line Test

A curve in the coordinate plane is the graph of a function if and only if no vertical line intersects the curve more than once.



Graph of a function



Not a graph of a function

Basic Functions

We introduce the basic functions that we will consider throughout the remainder of the semester.

- **polynomial functions**
  A polynomial function is a function of the form
  $$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$
  where n is a nonnegative integer and $a_0, a_1, \ldots, a_n$ are (real) constants with $a_n \neq 0$. The coefficient a_n is called the leading coefficient, and n is called the degree of the polynomial function. The largest possible domain of f is $\mathbb{R}$.

Examples Suppose a, b, c, and m are constants.

- Constant functions: $f(x) = c$ (graph is a horizontal line);
- Linear functions: $f(x) = mx + b$ (graph is a straight line);
- Quadratic functions: $f(x) = ax^2 + bx + c$ (graph is a parabola).

- **rational functions**

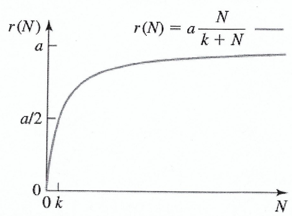
A rational function is the quotient of two polynomial functions $p(x)$ and $q(x)$: $f(x) = \dfrac{p(x)}{q(x)}$ for $q(x) \neq 0$.

Example The **Monod growth function** is frequently used to describe the per capita growth rate of organisms when the rate depends on the concentration of some nutrient and becomes saturated for large enough nutrient concentrations.

If we denote the concentration of the nutrient by N, then the per capita growth rate $r(N)$ is given by

$$r(N) = \frac{aN}{k+N}, \qquad N \geq 0$$

where a and k are positive constants.



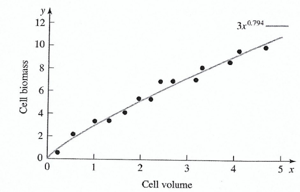
- **power functions**

A power function is of the form $f(x) = x^r$ where r is a real number.

Example Power functions are frequently found in "scaling relations" between biological variables (e.g., organ sizes).
Finding such relationships is the objective of **allometry**. For example, in a study of 45 species of unicellular algae, a relationship between cell volume and cell biomass was sought. It was found [see, Niklas (1994)] that

$$\text{cell biomass} \propto (\text{cell volume})^{0.794}$$

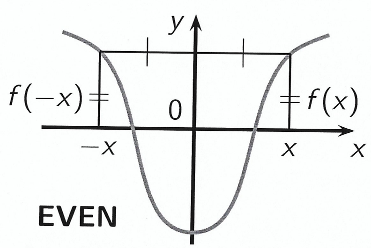
Most scaling relations are to be interpreted in a statistical sense; they are obtained by fitting a curve to data points. The data points are typically scattered about the fitted curve given by the scaling relation.



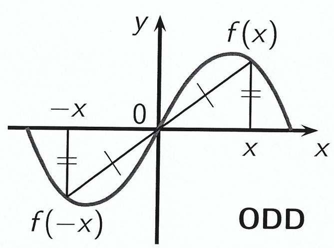
- **exponential and logarithmic functions**
- **trigonometric functions**

Even and Odd Functions

Let f be a function.
f is **even** if $f(-x) = f(x)$ for all x in the domain of f.
f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.



EVEN
Graph symmetric wrt y-axis.



ODD
Graph symmetric wrt $(0,0)$.

Example:

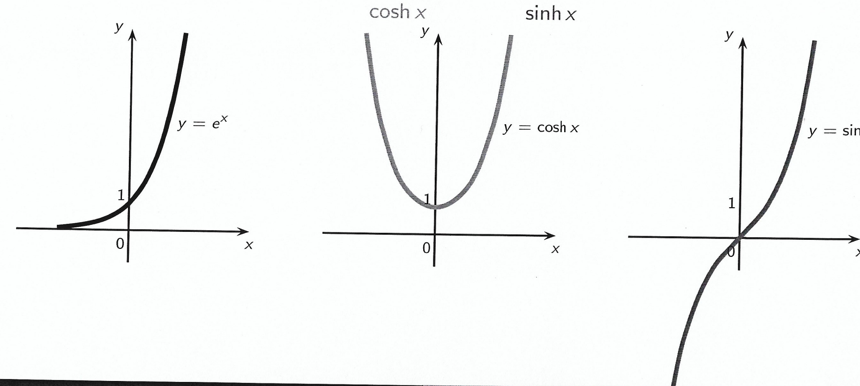
$y = \cos x$ is an **even** function; $y = \sin x$ is an **odd** function.

Curious/Amazing Fact!

Any function can be uniquely written as an even plus an odd function.

Example:

$$e^x = \underbrace{\frac{e^x + e^{-x}}{2}}_{\cosh x} + \underbrace{\frac{e^x - e^{-x}}{2}}_{\sinh x}$$



Combining functions

Let f and g be functions with domains A and B. We define new functions $f+g$, $f-g$, fg, and f/g as follows:

$(f+g)(x) = f(x) + g(x)$ Domain $A \cap B$

$(f-g)(x) = f(x) - g(x)$ Domain $A \cap B$

$(fg)(x) = f(x)g(x)$ Domain $A \cap B$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ Domain $\{x \in A \cap B \,|\, g(x) \neq 0\}$

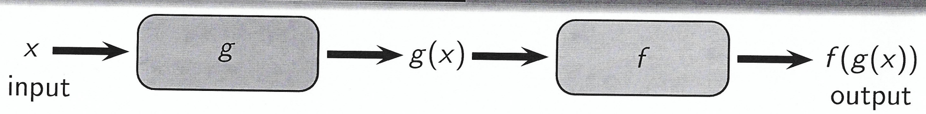
Composition of Functions

Given any two functions f and g, we start with a number x in the domain of g and find its image $g(x)$. If this number $g(x)$ is in the domain of f, we can then calculate the value of $f(g(x))$.

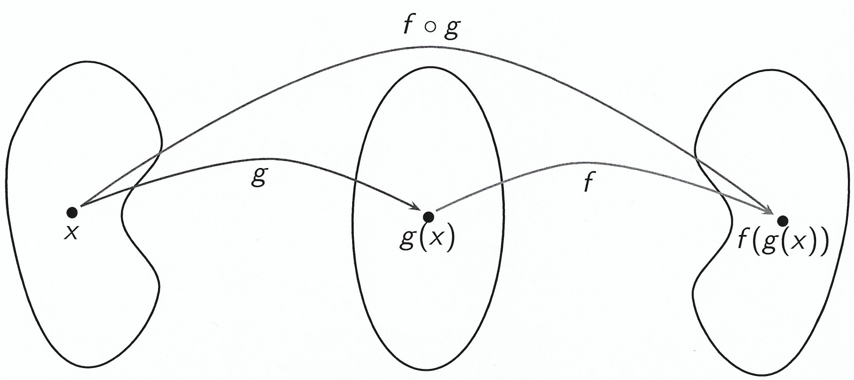
The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f. It is called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ (read: 'f composed with g' or 'f after g')

$$(f \circ g)(x) \stackrel{\text{def}}{=} f(g(x)).$$

WARNING: $f \circ g \neq g \circ f$.



Machine diagram of $f \circ g$



Arrow diagram of $f \circ g$

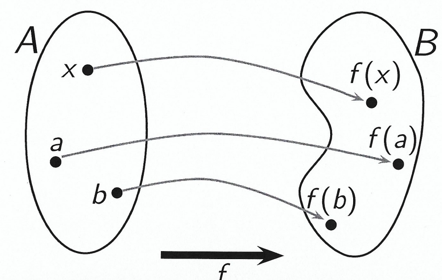
Definition of a One-One Function

A function f with domain A is called a **one-to-one function** if no two elements of A have the same image, that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2.$$

An equivalent way of writing the above condition is:

$$\text{If } f(x_1) = f(x_2), \text{ then } x_1 = x_2.$$

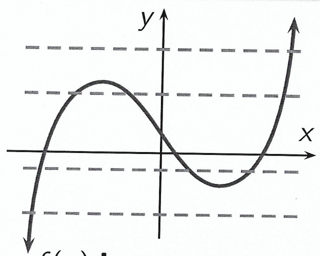


Horizontal Line Test

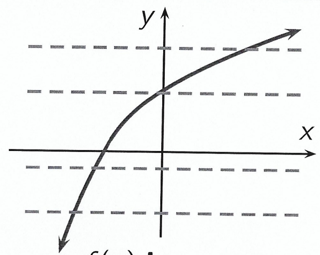
For functions that can be graphed in the coordinate plane, there is a useful criterion to determine whether a function is one-to-one or not.

Horizontal Line Test

A function is one-to-one
$\iff$
no horizontal line intersects its graph more than once.



$f(x)$ **is not** one-to-one



$f(x)$ **is** one-to-one

The Inverse of a Function

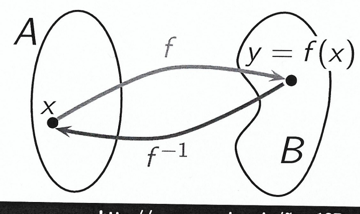
One-to-one functions are precisely those for which one can define a (unique) **inverse function** according to the following definition.

Definition of the Inverse of a Function

Let f be a one-to-one function with domain A and range B. Its **inverse function** f^{-1} has domain B and range A and is defined by

$$\boxed{f^{-1}(y) = x \iff f(x) = y,}$$

for any $y \in B$.



If f takes x to y, then f^{-1} takes y back to x. I.e., f^{-1} undoes what f does.
NOTE:
f^{-1} does NOT mean $\frac{1}{f}$.

Properties of Inverse Functions

Let $f(x)$ be a one-to-one function with domain A and range B. The inverse function $f^{-1}(y)$ satisfies the following "cancellation" properties:

$$\boxed{f^{-1}(f(x)) = x \quad \text{for every} \quad x \in A}$$

$$\boxed{f(f^{-1}(y)) = y \quad \text{for every} \quad y \in B}$$

Conversely, any function $f^{-1}(y)$ satisfying the above conditions is the inverse of $f(x)$.

Remark:
Typically we write functions in terms of x.
To do this, we need to interchange x and y in $x = f^{-1}(y)$.

How to find the Inverse of a One-to-One Function

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Graph of the Inverse Function

The principle of interchanging x and y to find the inverse function also gives us a method for obtaining the graph of f^{-1} from the graph of f. **The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.**

The picture on the right hand side shows the graphs of:

$$f(x) = \sqrt{x+4}$$

and

$$f^{-1}(x) = x^2 - 4, \ x \geq 0.$$