MA137 – Calculus 1 with Life Science Applications
Exponential and Logarithmic Functions
(Sections 1.2 and 1.3)

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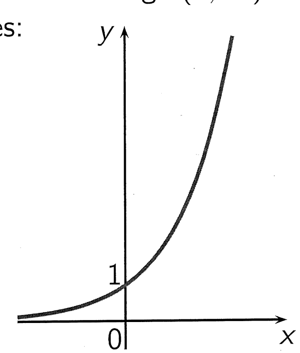
University of Kentucky

Exponential Functions

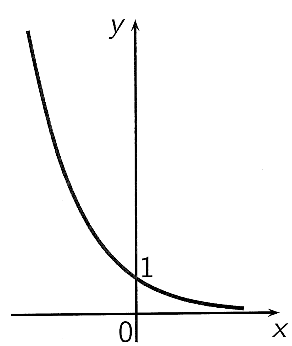
The **exponential function**

$$f(x) = a^x \qquad (a > 0,\ a \neq 1)$$

has domain $\mathbb{R}$ and range $(0, \infty)$. The graph of $f(x)$ has one of these shapes:



$f(x) = a^x$ for $a > 1$



$f(x) = a^x$
for $0 < a < 1$

Laws of Exponents

Let a and b be real numbers so that $a, b > 0$ and $a, b \neq 1$.

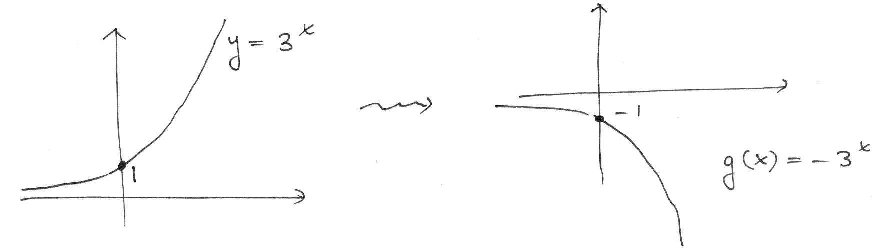
- $a^0 = 1$
- $a^u a^v = a^{u+v}$
- $\dfrac{a^u}{a^v} = a^{u-v}$ In particular, $\dfrac{1}{a^v} = \dfrac{a^0}{a^v} = a^{0-v} = a^{-v}$
- $(a^u)^v = a^{uv}$ In particular, $a^{1/n} = \sqrt[n]{a}$
- $(ab)^u = a^u b^u$ $\left(\dfrac{a}{b}\right)^u = \dfrac{a^u}{b^u}$

Example 1:

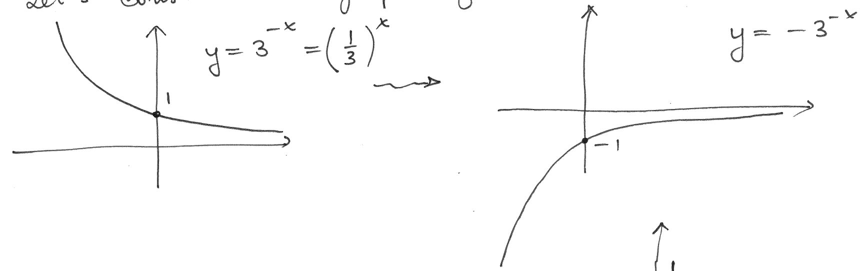
Use the graph of $f(x) = 3^x$ to sketch the graph of each function:

$g(x) = -3^x$

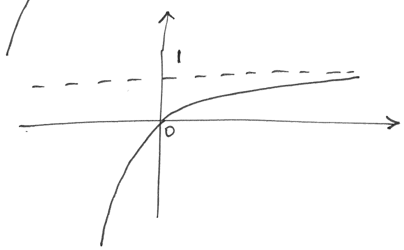
$h(x) = 1 - 3^{-x}$



Let's construct the graph of $h(x) = 1 - 3^{-x}$ in steps:



Finally, $y = 1 - 3^{-x}$



Why are exponential functions of interest?

Suppose that we study a population of 100 individuals and suppose that it grows annually at a 3% rate. Describe the population growth at time t.

(You can also consider \$ 100 in a bank growing annually at a 3%)

$P_0 = P(0) = 100$; growth rate $= 3\%$ or $r = 0.03$

$$P(1) = 100 + \underbrace{0.03 \cdot 100}_{\text{growth}} = 100 + 3 = 103 = 100\,(1.03)$$

$$P(2) = 103 + \underbrace{0.03 \cdot 103}_{\text{growth}} = 103\,(1 + 0.03) = 100\,(1.03)\,(1.03)$$

$$= 100\,(1.03)^2$$

In general $\boxed{P(t) = 100\,(1.03)^t}$ OR $\boxed{P(t) = P_0\,(1+r)^t}$

The formula for compounded interest is

$$\boxed{P(t) = P_0 \left(1 + \frac{r}{n}\right)^{nt}}$$

(see earlier discussion with $n = 1$)

where P_0 = initial principal

r = interest rate per year

n = number of times interest is compounded per year

t = number of years.

If n becomes very large ($\equiv$ interest is compounded continuously) the above formula becomes

$$\boxed{P(t) = P_0\, e^{rt}}$$

The Number 'e' (Euler's constant)

The most important base is the number denoted by the letter e.

The number e is defined as the value that $\left(1+\frac{1}{n}\right)^n$ approaches as n becomes very large.

Correct to five decimal places (note that e is an irrational number), $e \approx 2.71828$.

n	$\left(1+\frac{1}{n}\right)^n$
1	2.00000
5	2.48832
10	2.59374
100	2.70481
1,000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

The Natural Exponential Function

The Natural Exponential Function

The **natural exponential function** is the exponential function

$$f(x) = e^x$$

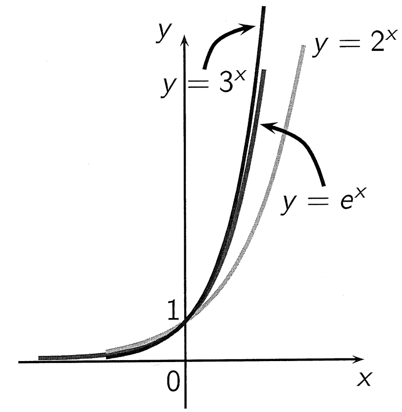
with base e. It is often referred to as *the* exponential function.

Note:
Sometimes we write

$$f(x) = \exp(x)$$

to denote the exponential function.

Since $2 < e < 3$, the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$.



Example 2:

When a certain drug is administered to a patient, the number of milligrams remaining in the patient's bloodstream after t hours is modeled by

$$D(t) = 50\,e^{-0.2t}.$$

How many milligrams of the drug remain in the patient's bloodstream after 3 hours?

(*) Notice that $D(0) = 50\,e^{-0.2\cdot 0} = 50\underbrace{e^0}_{1} = 50$

Thus 50 is the initial amount of drug administered to the patient.

(**) $D(3) = 50\,e^{-0.2\cdot 3} = 50\,e^{-0.6} \cong 27.44$ mg

(***) It would have been more interesting to ask: How long do we need to wait so that the blood stream of the patient only has 25 mg left of drug?

$25 = D(\bar{t}) = 50\,e^{-0.2\bar{t}} \iff \boxed{\frac{1}{2} = e^{-0.2\bar{t}}}$

How do we solve for $\bar{t}$?

Logarithmic Functions

Every exponential function $f(x) = a^x$, with $0 < a \neq 1$, is a one-to-one function (Horizontal Line Test). Thus, it has an inverse function, called the *logarithmic function with base a* and denoted by $\log_a x$.

Definition

Let a be a positive number with $a \neq 1$. The **logarithmic function** with base a, denoted by $\log_a$, is defined by

$$y = \log_a x \iff a^y = x.$$

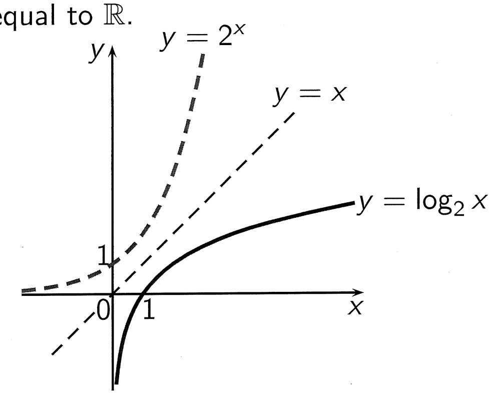
That is, $\log_a x$ is the exponent to which a must be raised to give x.

Properties of Logarithms

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a a^x = x$
4. $a^{\log_a x} = x$

Graphs of Logarithmic Functions

The graph of $f^{-1}(x) = \log_a x$ is obtained by reflecting the graph of $f(x) = a^x$ in the line $y = x$. Thus, the function $y = \log_a x$ is defined for $x > 0$ and has range equal to $\mathbb{R}$.



The point $(1, 0)$ is on the graph of $y = \log_a x$ (as $\log_a 1 = 0$) and the y-axis is a vertical asymptote.

Natural Logarithms

Of all possible bases a for logarithms, it turns out that the most convenient choice for the purposes of Calculus is the number e.

Definition

The logarithm with base e is called the **natural logarithm** and denoted:

$$\boxed{\ln x := \log_e x.}$$

We recall again that, by the definition of inverse functions, we have

$$y = \ln x \iff e^y = x.$$

Properties of Natural Logarithms

1. $\ln 1 = 0$
2. $\ln e = 1$
3. $\ln e^x = x$
4. $e^{\ln x} = x$

Common Logarithms

Another convenient choice of base for the purposes of the Life Sciences is the number 10.

Definition

The logarithm with base 10 is called the **common logarithm** and denoted:

$$\boxed{\log x := \log_{10} x.}$$

We recall again that, by the definition of inverse functions, we have

$$y = \log x \iff 10^y = x.$$

Properties of Natural Logarithms

1. $\log 1 = 0$
2. $\log 10 = 1$
3. $\log 10^x = x$
4. $10^{\log x} = x$

Laws of Logarithms

Since logarithms are 'exponents', the Laws of Exponents give rise to the Laws of Logarithms:

Laws of Logarithms

Let a be a positive number, with $a \neq 1$. Let A, B and C be any real numbers with $A > 0$ and $B > 0$.

1. $\log_a(AB) = \log_a A + \log_a B$;
2. $\log_a\left(\dfrac{A}{B}\right) = \log_a A - \log_a B$;
3. $\log_a(A^C) = C \log_a A$.

Proof of **Law 1.**: $\log_a(AB) = \log_a A + \log_a B$

Let us set

$$\log_a A = u \quad \text{and} \quad \log_a B = v.$$

When written in exponential form, they become

$$a^u = A \quad \text{and} \quad a^v = B.$$

Thus:

$$\begin{aligned} \underline{\log_a(AB)} &= \log_a(a^u\, a^v) \\ &= \log_a(a^{u+v}) \\ &\overset{\text{why?}}{=} u + v \\ &= \underline{\log_a A + \log_a B}. \end{aligned}$$

In a similar fashion, one can prove **2.** and **3.**

Expanding and Combining Logarithmic Expressions

Example 3:

Use the Laws of Logarithms to combine the expression

$$\log_a b + c \log_a d - r \log_a s - \log_a t$$

into a single logarithm.

$$\log_a b + c \log_a d - r \log_a s - \log_a t$$

$$= \left[\log_a b + \log_a (d^c)\right] - \left[\log_a (s^r) + \log_a t\right]$$

$$= \log_a (b d^c) - \log_a (s^r t)$$

$$= \underline{\underline{\log_a \left(\frac{b d^c}{s^r t}\right)}}$$

We used properties 1.-3. of Logarithms.

Change of Base

For some purposes, we find it useful to change from logarithms in one base to logarithms in another base. One can prove that:

$$\boxed{\log_b x = \frac{\log_a x}{\log_a b}.}$$

Proof: Set $y = \log_b x$. By definition, this means that $b^y = x$. Apply now $\log_a(\cdot)$ to $b^y = x$. We obtain

$$\log_a(b^y) = \log_a x \quad \rightsquigarrow \quad y \log_a b = \log_a x.$$

Thus

$$\log_b x = y = \frac{\log_a x}{\log_a b}.$$

Example: $\log_5 2 = \dfrac{\log 2}{\log 5} = \dfrac{\ln 2}{\ln 5} \approx 0.43068.$

Exponential Equations

An exponential equation is one in which the variable occurs in the exponent. For example,

$$3^{x+2} = 7.$$

We take the (either common or natural) logarithm of each side and then use the Laws of Logarithms to 'bring down the variable' from the exponent:

$$\log(3^{x+2}) = \log 7$$

$$\rightsquigarrow \quad (x+2)\log 3 = \log 7$$

$$\rightsquigarrow \quad x + 2 = \frac{\log 7}{\log 3}$$

$$\rightsquigarrow \quad x = \frac{\log 7}{\log 3} - 2 \approx -0.228756$$

Example 4: (Online Homework HW03, # 6)

Solve the given equation for x:

$$2^{5x-4} = 3^{10x-10}$$

$$2^{5x-4} = 3^{10x-10}$$

Take log of both sides (OR ln)

$$\log\left(2^{5x-4}\right) = \log\left(3^{10x-10}\right)$$

$$\iff (5x-4)\log 2 = (10x-10)\log 3$$

$$\iff (5\log 2)x - 4\log 2 = (10\log 3)x - 10\log 3$$

$$\iff (10\log 3)x - (5\log 2)x = 10\log 3 - 4\log 2$$

$$\iff [10\log 3 - 5\log 2]\,x = 10\log 3 - 4\log 2$$

$$\iff x = \frac{(10\log 3 - 4\log 2)}{(10\log 3 - 5\log 2)} = \frac{\log\left(\frac{3^{10}}{2^4}\right)}{\log\left(\frac{3^{10}}{2^5}\right)} \cong \underline{1.09216}$$

Logarithmic Equations

A logarithmic equation is one in which a logarithm of the variable occurs. For example,

$$\log_2(25 - x) = 3.$$

To solve for x, we write the equation in exponential form, and then solve for the variable:

$$25 - x = 2^3 \quad \rightsquigarrow \quad 25 - x = 8 \quad \rightsquigarrow \quad x = 17.$$

Alternatively, we raise the base, 2, to each side of the equation; we then use the Laws of Logarithms:

$$2^{\log_2(25-x)} = 2^3 \quad \rightsquigarrow \quad 25 - x = 2^3 \quad \rightsquigarrow \quad x = 17.$$

Example 5: (Online Homework HW03, # 5)

Solve the given equation for x:

$$\log_{10} x + \log_{10}(x + 21) = 2$$

$$\log_{10} x + \log_{10}(x+21) = 2$$

$$\iff \log_{10}\left[x(x+21)\right] = 2$$

$$\iff 10^{\log_{10}\left[x(x+21)\right]} = 10^2$$

$$\iff \boxed{x(x+21) = 100}$$

$$\iff x^2 + 21x - 100 = 0$$

$$\iff (x+25)(x-4) = 0$$

$$\iff x = -25,\ 4$$

HOWEVER, $\log_{10}(-25) + \log_{10}(-25+21) = 2$ does not make any sense! So $x = 4$ is the only solution.