MA137 – Calculus 1 with Life Science Applications
Applications: Exponential Growth and Decay
(Sections 1.2 and 1.3)

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Exponential Models of Population Growth

The formula for population growth of several species is the same as that for continuously compounded interest. In fact in both cases the rate of growth r of a population (or an investment) per time period is proportional to the size of the population (or the amount of an investment).

Exponential Growth Model

If n_0 is the initial size of a population that experiences **exponential growth**, then the population $n(t)$ at time t increases according to the model

$$\boxed{n(t) = n_0 e^{rt}}$$

where r is the relative rate of growth of the population (expressed as a proportion of the population).

Remark:

Biologists sometimes express the growth rate in terms of the **doubling-time** h, the time required for the population to double in size: $r = \dfrac{\ln 2}{h}$.

Proof: Indeed, from

$$2n_0 = n(h) = n_0 e^{rh}$$

we obtain

$$2 = e^{rh} \quad \rightsquigarrow \quad \ln 2 = rh \quad \rightsquigarrow \quad r = \frac{\ln 2}{h}.$$

Using the doubling-time h, we can also rewrite $n(t)$ as:

$$\boxed{n(t)} = n_0 e^{rt} = n_0 e^{\frac{\ln 2}{h} t} = n_0 e^{(t/h)\cdot \ln 2} = n_0 e^{\ln(2^{t/h})} \boxed{= n_0 2^{t/h}}.$$

Radioactive Decay

Radioactive substances decay by spontaneously emitting radiations. Also in this situation, the rate of decay is proportional to the mass of the substance and is independent of environmental conditions.

This is analogous to population growth, except that the mass of radioactive material *decreases*.

Radioactive Decay Model

If m_0 is the initial mass of a radioactive substance then the mass $m(t)$ remaining at time t is modeled by the function

$$\boxed{m(t) = m_0 e^{-rt}}$$

where r is the relative rate of decay of the radioactive substance.

Remark:

Physicists sometimes express the rate of decay in terms of the **half-life** h, the time required for half the mass to decay: $r = \dfrac{\ln 2}{h}$.

Proof: Indeed, from

$$\frac{1}{2} m_0 = m(h) = m_0 e^{-rh}$$

we obtain

$$\frac{1}{2} = e^{-rh} \quad \rightsquigarrow \quad \ln\frac{1}{2} = -rh \quad \rightsquigarrow \quad -\ln 2 = -rh \quad \rightsquigarrow \quad r = \frac{\ln 2}{h}.$$

Using the half-time h, we can also rewrite $m(t)$ as:

$$\boxed{m(t)} = m_0 e^{-rt} = m_0 e^{-\frac{\ln 2}{h} t} = m_0 e^{(-t/h)\cdot \ln 2} = m_0 e^{\ln(2^{-t/h})} \boxed{= m_0 \left(\frac{1}{2}\right)^{t/h}}.$$

Example 1: (Online Homework HW03, # 8)

A town has population 750 people at year $t = 0$.
Write a formula for the population, P, in year t if the town

(a) Grows by 70 people per year
(b) Grows by 12% per year
(c) Grows at a continuous rate of 12% per year.
(d) Shrinks by 14 people per year.
(e) Shrinks by 4% per year.
(f) Shrinks at a continuous rate of 4% per year.

(a) $P(t) = \underline{750 + 70t}$

(b) $P(t) = 750(1+0.12)^t = \underline{750(1.12)^t}$

(c) $P(t) = \underline{750\, e^{0.12t}}$

(d) $P(t) = \underline{750 - 14t}$

(e) $P(t) = 750(1-0.04)^t = \underline{750(0.96)^t}$

(f) $P(t) = \underline{750\, e^{-0.04t}}$

Example 2 (Frog Population):

The frog population in a small pond grows exponentially. The current population is 85 frogs, and the relative growth rate is 18% per year.

(a) Which function models the population after t years?
(b) Find the projected frog population after 3 years.
(c) When will the frog population reach 600?
(d) When will the frog population double?

(a) $n(t) = 85\, e^{0.18t}$

(b) $n(3) = 85\, e^{0.18(3)} = 85\, e^{0.54} \cong 145.86$

(c) We need to find $\bar{t}$ so that

$85\, e^{0.18\bar{t}} = n(\bar{t}) = 600 \Rightarrow e^{0.18\bar{t}} = \frac{600}{85}$

$\Rightarrow \ln e^{0.18\bar{t}} = \ln\left(\frac{600}{85}\right) \Rightarrow 0.18\bar{t} = \ln\left(\frac{600}{85}\right)$

$\therefore\ \bar{t} = \frac{\ln(600/85)}{0.18} \cong 10.857$ years

(d) Let h denote the doubling time. That is

$\cancel{85}\, e^{0.18h} = n(h) = 2 \cdot \cancel{85}$

$\Rightarrow \boxed{e^{0.18h} = 2} \Rightarrow \ln e^{0.18h} = \ln 2$

$\Rightarrow 0.18h = \ln 2 \Rightarrow h = \frac{\ln 2}{0.18} \cong 3.85$ years

Example 3: (Online Homework HW03, # 14)

Assume that the number of bacteria follows an exponential growth model: $P(t) = P_0 e^{kt}$. The count in the bacteria culture was 100 after 15 minutes and 1800 after 35 minutes.

(a) What was the initial size of the culture?
(b) Find the population after 105 minutes.
(c) How many minutes after the start of the experiment will the population reach 14,000?

(a) Using our information we have

$P(15) = P_0\, e^{15k} = 100$

$P(35) = P_0\, e^{35k} = 1800$

Thus $P_0 = \frac{100}{e^{15k}}$ and $P_0 = \frac{1800}{e^{35k}}$

$\Rightarrow \frac{100}{e^{15k}} = \frac{1800}{e^{35k}} \Rightarrow 100\, e^{35k} = 1800\, e^{15k}$

OR $\frac{e^{35k}}{e^{15k}} = \frac{1800}{100} \Rightarrow e^{35k-15k} = 18$

$\therefore \boxed{e^{20k} = 18} \Rightarrow 20k = \ln 18 \Rightarrow \boxed{k = \frac{\ln 18}{20}}$

Thus $k = 0.14518$

What about P_0? Since $P_0 = \frac{100}{e^{15k}}$, for example, we obtain $P_0 = \dfrac{100}{e^{15\left(\frac{\ln 18}{20}\right)}} = \dfrac{100}{e^{3/4 (\ln 18)}}$

$= \dfrac{100}{e^{\ln(18^{3/4})}} = \boxed{\dfrac{100}{18^{3/4}}} \cong \underline{11.4431508}$

(b) $\underline{P(t)} = \dfrac{100}{18^{3/4}} \cdot e^{\left[\frac{\ln 18}{20}\right] t} = \dfrac{100}{18^{3/4}} e^{\left[t/20 \ln 18\right]} =$

$= \dfrac{100}{18^{3/4}} \cdot e^{\ln\left(18^{t/20}\right)} = \dfrac{100}{18^{3/4}} \cdot 18^{t/20}$

$= 100 \cdot 18^{(t/20 - 3/4)} = \underline{100 \cdot 18^{\left(\frac{t-15}{20}\right)}}$

$\therefore \underline{P(105)} = 100 \cdot 18^{\left(\frac{105-15}{20}\right)} = \underline{100 \cdot 18^{4.5}}$

$\cong \underline{\underline{44{,}537{,}544.88}}$

(c) Finally,

$14{,}000 = 100 \cdot 18^{\left(\frac{\bar{t}-15}{20}\right)}$

$\Longrightarrow 140 = 18^{\frac{\bar{t}-15}{20}}$

$\Longrightarrow \ln(140) = \left(\dfrac{\bar{t}-15}{20}\right) \cdot \ln(18)$

$\Longrightarrow 20 \ln(140) = \bar{t} \cdot \ln(18) - 15 \cdot \ln(18)$

$\Longrightarrow \dfrac{20 \ln(140) + 15 \ln(18)}{\ln(18)} = \bar{t}$

$\cong \underline{49.1938189}$ minutes

Example 4:

The mass $m(t)$ remaining after t days from a 40-g sample of thorium-234 is given by:

$$m(t) = 40e^{-0.0277\,t}.$$

(a) How much of the sample will be left after 60 days?

(b) After how long will only 10-g of the sample remain?

(a) $m(60) = 40\, e^{-0.0277 \cdot 60} \cong \underline{7.59036 \text{ mg}}$

(b) $\underline{10} = m(\bar{t}) = \underline{40\, e^{-0.0277 \cdot \bar{t}}}$

we need to find $\bar{t}$.

$\Longrightarrow \dfrac{1}{4} = e^{-0.0277 \bar{t}}$

$\Longrightarrow \ln(1/4) = \ln e^{-0.0277 \bar{t}} = -0.0277 \bar{t}$

$\therefore \bar{t} = \dfrac{\ln(1/4)}{-0.0277} = \dfrac{\cancel{\ln 1}^{0} - \ln 4}{-0.0277} = \dfrac{\ln 4}{0.0277}$

$\cong \underline{50.046 \text{ days}}$

From Neuhauser's Textbook, p. 27

[...] Carbon 14 is formed high in the atmosphere. It is radioactive and decays into nitrogen (N^{14}).

There is an equilibrium between atmospheric carbon 12 (C^{12}) and carbon 14 (C^{14}) — a ratio that has been relatively constant over a fairly long period.

When plants capture carbon dioxide (CO_2) molecules from the atmosphere and build them into a product (such as cellulose), the initial ratio of C^{14} to C^{12} is the same as that in the atmosphere.

Once the plants die, however, their uptake of CO_2 ceases, and the radioactive decay of C^{14} causes the ratio of C^{14} to C^{12} to decline.

Because the law of radioactive decay is known, the change in ratio provides an accurate measure of the time since the plants death.

Example 5: (Neuhauser, Problem # 65, p.38)

The half-life of C^{14} is 5730 years. Suppose that wood found at an archeological excavation site contains about 35% as much C^{14} (in relation to C^{12}) as does living plant material.

Determine when the wood was cut.

By a previous discussion

$$\boxed{m(t) = m_0 \left(\tfrac{1}{2}\right)^{t/5730}}$$

Hence we are seeking $\bar{t}$ such that

$$\underline{0.35\, m_0} = m(\bar{t}) = \underline{m_0 \left(\tfrac{1}{2}\right)^{\bar{t}/5730}}$$

$$\therefore\quad 0.35 = \left(\tfrac{1}{2}\right)^{\bar{t}/5730} \Longrightarrow \ln(0.35) = \frac{\bar{t}}{5730} \ln(1/2)$$

$$\therefore\quad \bar{t} = \frac{5730 \ln(0.35)}{\ln(1/2)} = \frac{5730 \ln(0.35)}{-\ln(2)} =$$

$$= \frac{5730 \cdot (-1) \ln(0.35)}{\ln(2)} = \frac{5730 \ln\left(\frac{1}{0.35}\right)}{\ln(2)} \cong \underline{8678.50428 \text{ years}}$$

Newton's Law of Cooling

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that the temperature difference is not too large.

Using Calculus, the following model can be deduced from this law:

The Model

If D_0 is the initial temperature difference between an object and its surroundings, and if its surroundings have temperature T_S, then the temperature of the object at time t is modeled by the function

$$\boxed{T(t) = T_S + D_0 e^{-kt}}$$

where k is a positive constant that depends on the object.

Example 6 (Cooling Turkey):

A roasted turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F.

(a) If the temperature of the turkey is 150°F after half an hour, what is its temperature after 45 minutes?

(b) When will the turkey cool to 100°F?

(a) $D_0 = 185 - 75 = 110$

hence the temperature of the turkey is given by

$$T(t) = 75 + 110\, e^{-kt}$$

So $T(30) = 75 + 110\, e^{-30k} = 150$

$$\Longrightarrow e^{-30k} = \frac{150-75}{110} \Longrightarrow k = \frac{\ln(75/110)}{-30}$$

$\therefore\ k \cong 0.01276$

Hence $\boxed{T(t) = 75 + 110\, e^{-0.01276\, t}}$

(b) Check that $T(\bar{t}) = 100 \Longrightarrow$

$\underline{\bar{t} = 116 \text{ minutes}}$ (almost 2 hours)

Interested in Forensic Pathology?

Newton's Law of Cooling is used in **homicide investigations** to determine the time of death. Immediately following death, the body begins to cool (its normal temperature is 98.6°F). It has been experimentally determined that the constant in Newton's Law of Cooling is $k \approx 0.1947$, assuming time is measured in hours.