MA137 – Calculus 1 with Life Science Applications
Discrete-Time Models
Sequences and Difference Equations
(Sections 2.1 and 2.2)

Department of Mathematics
University of Kentucky

What are sequences?

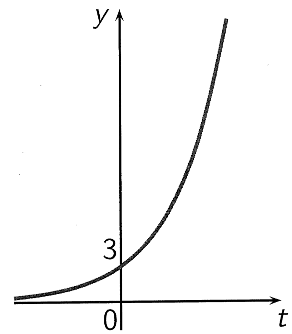
So far we have studied real valued functions whose domain consists of the real numbers, say:

$$f : \mathbb{R} \longrightarrow \mathbb{R}.$$

For example, consider the function

$$f(t) = 3 \cdot 2^t.$$

The graph of f looks like:



More generally, we have considered functions of the form

$$P(t) = P_0(1+r)^t,$$

where r is a positive real number ($r \equiv$ growth rate).

Sometimes it makes sense to change the domain of the function to the nonnegative integers $\mathbb{N} = \{0, 1, 2, 3, \dots\}$

$$f : \mathbb{N} \longrightarrow \mathbb{R}, \qquad n \mapsto f(n).$$

For example, $f(n) = 3 \cdot 2^n$ with $n \in \mathbb{N}$.

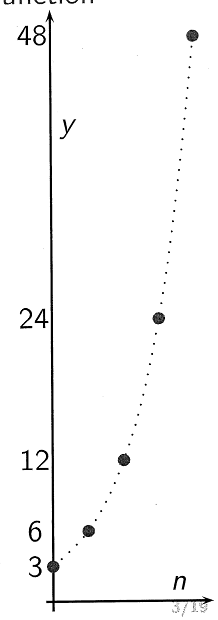
A table is a useful tool to illustrate this function

n	0	1	2	3	4	$\cdots$
$3 \cdot 2^n$	3	6	12	24	48	$\cdots$

The graph is useful too!

Because the domain consists of nonnegative integers, the graph consists of isolated points with coordinates

$$(0, f(0)) \quad (1, f(1)) \quad (2, f(2)) \quad (3, f(3)) \quad (4, f(4)) \quad \cdots$$



Note: we should not have connected the isolated points with the dotted curve. Please disregard it!!

Definition and Notation

Definition (Sequence/Notation)

We can write the function

$$f : \mathbb{N} \longrightarrow \mathbb{R}, \qquad n \mapsto f(n)$$

as a list of numbers $f_0,\ f_1,\ f_2,\ f_3,\ \dots,$ where $f_n = f(n)$.

We refer to this list as a **sequence**.

We write $\{f_n \mid n \in \mathbb{N}\}$ (or $\{f_n\}$ for short) to denote the entire sequence.

We list the values of the sequence $\{f_n\}$ in order of increasing n

$$f_0,\ f_1,\ f_2,\ f_3,\ \dots$$

Remark: Instead of 'f' we often use the letters 'a' or 'b' or 'c' ... to denote sequences.

For example: $\quad a_n = \dfrac{n}{n+1} \qquad b_n = \dfrac{(-1)^n}{(n+1)^2} \qquad c_n = 3 \cdot 2^n$

Example 1:

Find a general formula for the general term a_n for each of the following sequences starting with a_0:

(a) $0, 1, 4, 9, 16, 25, 36, 49, \ldots$

(b) $1, -1, 1, -1, 1, -1, \ldots$

(c) $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, \ldots.$

Repeat this problem starting this time with a_1.

(a) Consider $0, 1, 4, 9, 16, 25, 36, 49, \ldots$

these are all squares of numbers.

We want them to be labeled as

$a_0 = 0, \quad a_1 = 1, \quad a_2 = 4, \quad a_3 = 9, \quad a_4 = 16, \ldots$

thus $\boxed{a_n = n^2}$ is the nth term of the sequence

(b) We want: $a_0 = 1, \quad a_1 = -1, \quad a_2 = 1, \quad a_3 = -1, \ldots$

So we have $\boxed{a_n = (-1)^n}$ for all $n \in \mathbb{N}$

(c) We want: $a_0 = 1, \quad a_1 = -\frac{1}{2}, \quad a_2 = \frac{1}{4}, \quad a_3 = -\frac{1}{8}$

$a_4 = \frac{1}{16}$, etc... Notice that all denominators are powers of 2; there is an alternating sign: $\boxed{a_n = \left(-\frac{1}{2}\right)^n}$

Repeat:

(a) this time we want: $a_1 = 0, \quad a_2 = 1, \quad a_3 = 4,$

$a_4 = 9, \quad a_5 = 16, \ldots$ Thus we need to shift the integers:

$$a_n = (n-1)^2 \qquad \text{for } n = 1, 2, 3, 4, \ldots$$

(b) We want: $a_1 = 1, \quad a_2 = -1, \quad a_3 = 1, \quad a_4 = -1, \ldots$

again we shift the integers:

$$a_n = (-1)^{n-1} \quad \text{or} \quad a_n = (-1)^{n+1} \qquad n = 1, 2, 3, 4, \ldots$$

(c) We want: $a_1 = 1, \quad a_2 = -\frac{1}{2}, \quad a_3 = \frac{1}{4}, \quad a_4 = -\frac{1}{8}, \ldots$

$$a_n = \left(-\frac{1}{2}\right)^{n-1} \qquad n = 1, 2, 3, 4, \ldots$$

Example 2:

Consider the sequence given by

$$a_n = 2 + \frac{(-1)^n}{n} \qquad n > 1.$$

List the first six terms of the sequence and plot them on the Cartesian plane.

$$a_n = 2 + \frac{(-1)^n}{n} \qquad n > 1$$

notice that the expression does <u>not</u> make sense for $n=0$.

$$a_1 = 2 + \frac{(-1)^1}{1} = 2 - 1 = \underline{1}$$

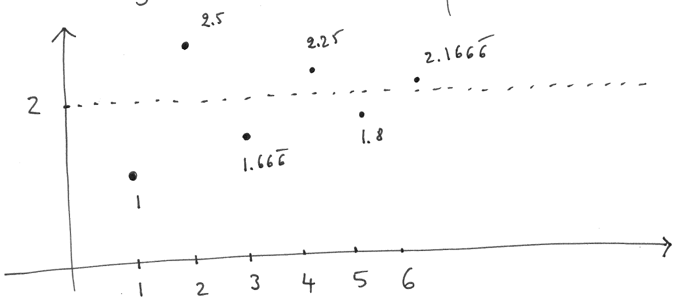
$$a_2 = 2 + \frac{(-1)^2}{2} = \underline{2.5}$$

$$a_3 = 2 + \frac{(-1)^3}{3} = 2 - \tfrac{1}{3} = \underline{1.66\overline{6}}$$

$$a_4 = 2 + \frac{(-1)^4}{4} = \underline{2.25}$$

$$a_5 = 2 + \frac{(-1)^5}{5} = \underline{1.8}$$

$$a_6 = 2 + \frac{(-1)^6}{6} = 2.166\overline{6}$$



Recursions (or Recursive Sequences)

The exponential growth model we considered earlier

$$P_n = 3 \cdot 2^n$$

is an example of a sequence. Explicitly, we have

$$P_0 = 3, \quad P_1 = 6, \quad P_2 = 12, \quad P_3 = 24, \quad P_4 = 48, \quad \cdots$$

It is not difficult to observe that this sequence of numbers describes quantities that double after each unit of time.
More explicitly, we can write

$$P_1 = 2P_0, \quad P_2 = 2P_1, \quad P_3 = 2P_2, \quad P_4 = 2P_3, \quad \cdots$$

We can summarize the above facts into a single expression. I.e.,

$$P_{n+1} = 2P_n$$

this expression gives a rule that is applied repeatedly to go from one time step (the nth) to the next one (the $(n+1)$st).
Such an expression is called a **recursion**.

Example 3:

(a) List the first five terms of the recursively define sequence

$$a_0 = 1 \qquad a_{n+1} = (n+1)a_n.$$

Do you see something familiar?

(b) List the first five terms of the recursively define sequence

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = 1 + \frac{1}{a_n}.$$

Do you see something familiar?

Caution: While it is easy to compute terms in a recursive relation, there are 2 issues:

- In order to find a_{100}, we have to compute the previous 99 terms.
- We may not get a feeling for what will eventually happen.

$a_0 = 1 \qquad a_{n+1} = (n+1)\,a_n \qquad n = 0, 1, 2, 3, \ldots$

when $n=0$: $a_1 = 1 \cdot a_0 = 1$

when $n=1$: $a_2 = (1+1)\,a_1 = 2 \cdot 1 = 2!$

when $n=2$: $a_3 = (2+1)\,a_2 = 3 \cdot a_2 = 3 \cdot 2 \cdot 1 = 3!$

when $n=3$: $a_4 = (3+1)\,a_3 = 4 \cdot a_3 = 4 \cdot 3 \cdot 2 \cdot 1 = 4!$

when $n=4$: $a_5 = (4+1)\,a_4 = 5 \cdot a_4 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$

In general the explicit form for the sequence is: $\boxed{a_n = n!}$ for $n = 0, 1, 2, \ldots$

$a_1 = 1 \qquad a_{n+1} = 1 + \frac{1}{a_n}$ for $n = 1, 2, 3, 4, 5, \ldots$

when $n=1$ $\quad a_2 = 1 + \frac{1}{a_1} = 1 + \frac{1}{1} = \underline{\underline{2}}$

when $n=2$ $\quad a_3 = 1 + \frac{1}{a_2} = 1 + \frac{1}{2} = \underline{\underline{3/2}} \cong 1.5$

when $n=3$ $\quad a_4 = 1 + \frac{1}{a_3} = 1 + \frac{1}{3/2} = 1 + \frac{2}{3} = \underline{\underline{\frac{5}{3}}} \cong 1.\overline{666}$

when $n=4$ $\quad a_5 = 1 + \frac{1}{a_4} = 1 + \frac{1}{5/3} = 1 + 3/5 = \underline{\underline{\frac{8}{5}}} \cong 1.6$

when $n=5$ $\quad a_6 = 1 + \frac{1}{a_5} = 1 + \frac{1}{8/5} = 1 + \frac{5}{8} = \underline{\underline{\frac{13}{8}}} \cong 1.625$

this sequence is given by the quotient of 2 consecutive Fibonacci's numbers

when $n \longrightarrow \infty$ this ratio tends to $1.618 \approx \frac{1+\sqrt{5}}{2}$

$\boxed{\text{GOLDEN RATIO}}$

Spreedsheets to Calculate Recursive Sequences

Using a spreadsheet it is possible to quickly calculate many terms in any sequence that is defined by a recurrence equation. We will explain how to do this calculation, using the specific recursive sequence of Example 3(b), that is:

$$a_1 = 1, \quad a_{n+1} = 1 + \frac{1}{a_n} \qquad (*)$$

We will use the column **A** of the spreadsheet to store the values of the index n for each term in the sequence and column **B** to store the values of the sequence a_n. Use the cells **A1** and **B1** to label the columns **n** and **a_n** respectively, and cells **A2** and **B2** to enter the index (1) and value (1) for the first term (a_1) in the sequence. To generate the next row we need to use the recursion equation (*).

In cell **A3** enter 2 (the index) and in cell **B3** enter **=1+1/B2**, as shown in the picture below

	A	B
1	n	a_n
2	1	1
3	2	=1+1/B2
4		

The value of **B3** (a_2) will then be computed from the value of **B2** (a_1) as the recurrence equation requires. We can then use the spreadsheet's **Autofill** command to generate the further terms in the sequence. Select the last row of your table (i.e., the cells **A3** and **B3**). When you select them, these two cells will be highlighted and surrounded by a colored outline. In the bottom right corner of the outline is a small colored square.

	A	B
1	n	a_n
2	1	1
3	2	2
4		

Click and hold on the square, and then drag down several rows, as shown below

	A	B
1	n	a_n
2	1	1
3	2	2
4	3	1.5
5	4	1.666667
6	5	1.6
7	6	1.625
8	7	1.615385
9	8	1.619048
10	9	1.617647
11	10	1.618182
12	11	1.617978
13		

The spreadsheet will automatically fill the new rows using the recursion formula. Specifically it fills **A4** with the index 3, **A5** with 4, and so on. More importantly, it will put the formula **=1+1/B3** in **B4**. Since **B3** holds the value a_2, **B4** will hold the value $1 + 1/a_2$, which is our formula for a_3; **B5** gets filled with the formula **=1+1/B4**, which gives $1 + 1/a_3$ the formula for a_4. The number of terms that are calculated in the sequence is the number of rows that we pull down the fill-box.

Example 4: (Online Homework HW05, # 8)

(a) Find a recursive definition for the sequence $9, 11, 13, 15, 17, \ldots$ Assume the first term in the sequence is indexed by $n = 1$.

(b) Find a closed formula for the sequence $9, 11, 13, 15, 17, \ldots$ Assume the first term in the sequence is indexed by $n = 1$.

$9, 11, 13, 15, 17, \ldots$

every number is obtained from the previous one by adding two:

$a_1 = 9, \quad a_2 = 11 = 9+2 = 9+2(1), \quad a_3 = 13 = 9+4 = 9+2(2), \quad a_4 = 15 = 9+6 = 9+2(3), \quad a_5 = 17 = 9+8 = 9+2(4)$

Recursive: $\boxed{a_1 = 9 \qquad a_{n+1} = a_n + 2}$ $\quad n = 1, 2, 3, \ldots$

Explicit: $\boxed{a_n = 9 + 2(n-1)}$ $\quad n = 1, 2, 3, 4, \ldots$

Recap

We gave two descriptions of sequences: explicit and recursive.

- An **explicit description** is of the form $a_n = f(n)$, $n = 0, 1, 2, \ldots$ where $f(n)$ is a function of n.
- A **recursive description** is of the form $a_{n+1} = g(a_n)$, $n = 0, 1, 2, \ldots$ where $g(a_n)$ is a function of a_n.

Remark 1:
In the above situation the value of a_{n+1} depends only on the value one time step back, namely, a_n. In this case the recursion is called a **first-order recursion** .

Remark 2:
The sequence defined by

$$a_0 = 1, \quad a_1 = 1, \quad a_{n+2} = a_n + a_{n+1} \quad \text{for} \quad n = 0, 1, 2, \ldots$$

is an example of a **second-order recursion**.

Recursive Sequences in the Life Sciences

Recursive sequences (or **difference equations**) are often used in biology to model, for example, cell division and insect populations.

In this biological context we usually replace n by t, to denote time.

If we think of t as the current time, then $t + 1$ is one unit of time into the future. We also use N_t to denote the population size.

Thus a first-order difference equation modeling population size has the form

$$N_{t+1} = f(N_t) \qquad t = 0, 1, 2, 3, \ldots$$

In this context we call f an **updating function** because f 'updates' the population from N_t to N_{t+1}.

Malthusian (or Exponential) Growth Model

One of our earlier examples can be rewritten as

$$N_{t+1} = 2N_t \quad N_0 = 3 \qquad \text{or} \qquad N_t = 3 \cdot 2^t.$$

This example is a special case of the so called **Malthusian Growth Model**, named after Thomas Malthus (1766-1834):

$$N_{t+1} = (1 + r)N_t$$

which says that the next generation is proportional to the population of the current generation.

It is typical to set $R = 1 + r$ so that the recursion becomes

$$N_{t+1} = RN_t.$$

This recursion has the following explicit form

$$N_t = N_0 R^t.$$

Hence the name of Exponential Growth Model.

Example 5: (Online Homework HW05, # 11)

(a) A population of herbivores satisfies the growth equation $y_{n+1} = 1.05y_n$, where n is in years. If the initial population is $y_0 = 6,000$, then determine the explicit expression of the population.

(b) A competing group of herbivores satisfies the growth equation $z_{n+1} = 1.06z_n$ If the initial population is $z_0 = 3,200$, then determine how long it takes for this population to double.

(c) Find when the two populations are equal.

(a) $y_n = 6,000\,(1.05)^n$

(b) $z_n = 3,200\,(1.06)^n$

we want to know n such that

$$\underline{3,200\,(1.06)^n} = z_n = \underline{2 \cdot 3,200}$$

i.e. we want $(1.06)^n = 2$

take log (or ln) of both sides

$$\log(1.06)^n = \log(2) \implies n = \frac{\log 2}{\log(1.06)} \cong \underline{11.895}$$

(c) We want to find n such that the two populations are equal:

$$6,000\,(1.05)^n = 3,200\,(1.06)^n$$

Rewrite as:

$$\frac{6,000}{3,200} = \frac{(1.06)^n}{(1.05)^n} \quad \text{OR} \quad \frac{15}{8} = \left(\frac{1.06}{1.05}\right)^n$$

Take log (or ln) of both sides

$$\log\left(\frac{15}{8}\right) = \log\left[\left(\frac{1.06}{1.05}\right)^n\right]$$

$$\implies n \log\left(\frac{1.06}{1.05}\right) = \log(15/8)$$

$$\therefore\ n = \frac{\log(15/8)}{\log\left(\frac{1.06}{1.05}\right)} \cong \underline{66.3177}$$

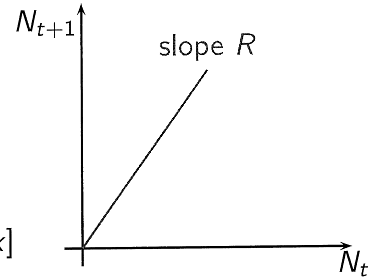
Visualizing Recursions

We can visualize recursions by plotting N_t on the horizontal axis and N_{t+1} on the vertical axis. Since $N_t \geq 0$ for biological reasons, we restrict the graph to the first quadrant.

The exponential growth recursion

$$N_{t+1} = RN_t$$

is then a straight line through the origin with slope R.
[i.e., $N_{t+1} = f(N_t)$, where $f(x) = Rx$]



For any current population size N_t, the graph allows us to find the population size in the next time step, namely, N_{t+1}.

Unless we label the points according to the corresponding t-value, we would not be able to tell at what time a point (N_t, N_{t+1}) was realized. We say that **time is implicit in this graph**.

The hallmark of exponential growth is that the ratio of successive population sizes, N_t/N_{t+1}, is constant. More precisely, it follows from $N_{t+1} = RN_t$ that

$$\frac{N_t}{N_{t+1}} = \frac{1}{R}$$

If the population consists of annual plants, we can interpret the ratio N_t/N_{t+1} as the **parent-offspring ratio**.

If this ratio is constant, parents produce the same number of offspring, regardless of the current population density. Such growth is called **density independent**.

When $R > 1$, the parent-offspring ratio, is less than 1, implying that the number of offspring exceeds the number of parents. This model yields then an ever-increasing population size. It eventually becomes **biologically unrealistic**, since any population will sooner or later experience food or habitat limitations that will limit its growth.

Below is the graph of the parent-offspring ratio N_t/N_{t+1} as a function of N_t when $N_t > 0$.