MA 137 – Calculus 1 with Life Science Applications
Discrete-Time Models
Sequences and Difference Equations: **Limits**
(Section 2.2)

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Long-Term Behavior

When studying populations over time, we are often interested in their long-term behavior.

Specifically, if N_t is the population size at time t, $t = 0, 1, 2, \ldots$, we want to know how N_t behaves as t increases, or, more precisely, as t tends to infinity.

Using our general setup and notation, we want to know the behavior of a_n as n tends to infinity and use the shorthand notation

$$\lim_{n \longrightarrow \infty} a_n$$

which we read as 'the limit of a_n as n tends to infinity.'

Definition and Notation

Definition (Informal)

We say that the limit as n tends to infinity of a sequence a_n is a number L, written as $\lim_{n \longrightarrow \infty} a_n = L$, if we can make the terms a_n as close to L as we like by taking n sufficiently large.

Definition (Formal)

The sequence $\{a_n\}$ has a limit L, written as $\lim_{n \longrightarrow \infty} a_n = L$, if, for any given any number $d > 0$, there is an integer N so that

$$|a_n - L| < d$$

whenever $n > N$.

If the limit exists, the sequence **converges** (or is **convergent**).

Otherwise we say that the sequence **diverges** (or is **divergent**).

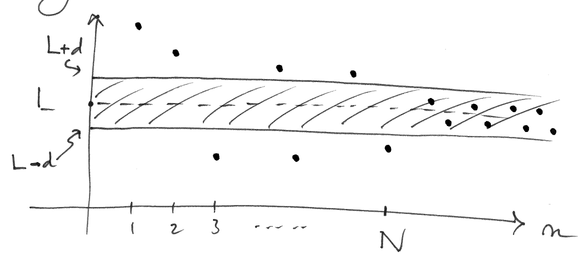
The informal definition of limit says that we can make the terms a_n as close to the limit L as we like.

The formal definition says that for any given number $d > 0$ there exists an integer N so that $|a_n - L| < d$ whenever $n > N$.

If we rework it out we have

$|a_n - L| < d \iff -d < a_n - L < d \iff \boxed{L-d < a_n < L+d}$

geometrically, this means that if we plot the graph of the sequence in the Cartesian plane we have the following situation:



any number d defines a strip in the plane about the line L of amplitude $2d$.

The points (n, a_n) are perhaps not in that strip for $n \leq N$ ... however for $n > N$ all the points (n, a_n) are in the strip.

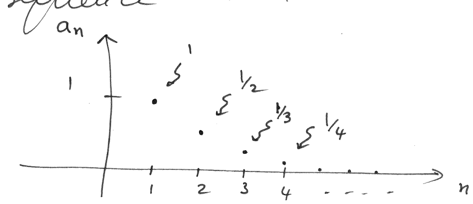
If we make "d" smaller, i.e. the strip is smaller, we can choose N larger.

Example 1:

Let $a_n = \dfrac{1}{n}$ for $n = 1, 2, 3, \ldots$.

Show that $\lim\limits_{n \longrightarrow \infty} \dfrac{1}{n} = 0$

Intuitively, the $\lim\limits_{n \to \infty} \frac{1}{n}$ is equal to 0 because if we plot the points corresponding to this sequence in the cartesian plane we have



those points get closer and closer to the n-axis.

Formally, for any $d > 0$ we need to find N such that $|a_n - L| < d$ whenever $n > N$.

But: $\left|\frac{1}{n} - 0\right| < d \iff \frac{1}{n} < d$ (as $n > 0$)

$\iff \frac{1}{d} < n$. So choose $\boxed{N = \frac{1}{d}}$.

Example 2:

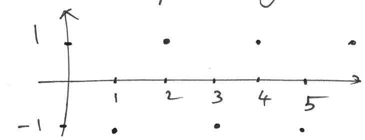
Let $a_n = (-1)^n$ for $n = 0, 1, 2, \ldots$.

Show that $\lim\limits_{n \longrightarrow \infty} (-1)^n$ does not exist.

What about the limit of the sequence $b_n = \cos(\pi n)$?

$\lim\limits_{n \to \infty} (-1)^n$ = does not exist

If we plot the points corresponding to this sequence we get



This means that for consecutive values of the index, say n and $n+1$ the difference $a_n - a_{n+1}$ is in absolute value always $\underline{\underline{2}}$ ... even if n goes to infinity. The do not get closer to a common value.

Limit Laws

The operations of arithmetic, namely, addition, subtraction, multiplication, and division, all behave reasonably with respect to the idea of getting closer to as long as nothing illegal happens.

This is summarized by the following laws:

If $\lim\limits_{n \longrightarrow \infty} a_n$ and $\lim\limits_{n \longrightarrow \infty} b_n$ exist and c is a constant, then

1. $\lim\limits_{n \longrightarrow \infty} (a_n + b_n) = (\lim\limits_{n \longrightarrow \infty} a_n) + (\lim\limits_{n \longrightarrow \infty} b_n)$
2. $\lim\limits_{n \longrightarrow \infty} (c\, a_n) = c\,(\lim\limits_{n \longrightarrow \infty} a_n)$
3. $\lim\limits_{n \longrightarrow \infty} (a_n b_n) = (\lim\limits_{n \longrightarrow \infty} a_n)(\lim\limits_{n \longrightarrow \infty} b_n)$
4. $\lim\limits_{n \longrightarrow \infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n \longrightarrow \infty} a_n}{\lim\limits_{n \longrightarrow \infty} b_n}$, provided $\lim\limits_{n \longrightarrow \infty} b_n \neq 0$

Example 3:

Find $\lim_{n\to\infty} \frac{n(1-3n^2)}{n^3+1}$.

Find $\lim_{n\to\infty} \frac{n}{n^2+1}$.

(a) $\lim_{n\to\infty} \frac{n(1-3n^2)}{n^3+1}$ = using the limit laws

$$= \frac{(\lim_{n\to\infty} n)(\lim_{n\to\infty} 1-3n^2)}{\lim_{n\to\infty}(n^3+1)} = \text{etc....}$$

$$= \frac{\infty(-\infty)}{\infty} = \text{which is not defined.}$$

However, notice that $\lim_{n\to\infty} \frac{1}{n} = 0$

$$\Longrightarrow \boxed{\lim_{n\to\infty} \frac{1}{n^p} = 0} \quad \text{for any } p>1$$

Thus we can rewrite our original limit as

$$\lim_{n\to\infty} \frac{n(1-3n^2)}{n^3+1} = \lim_{n\to\infty} \frac{n-3n^3}{n^3+1} =$$

$$= \lim_{n\to\infty} \frac{(n-3n^3)\cdot\frac{1}{n^3}}{(n^3+1)\cdot\frac{1}{n^3}} = \lim_{n\to\infty} \frac{\left(\frac{1}{n^2}-3\right)}{\left(1+\frac{1}{n^3}\right)}$$

use now the properties of limits:

$$= \frac{\lim_{n\to\infty}\left(\frac{1}{n^2}-3\right)}{\lim_{n\to\infty}\left(1+\frac{1}{n^3}\right)} = \frac{\left[\lim_{n\to\infty}\left(\frac{1}{n^2}\right)\right]-\left[\lim_{n\to\infty} 3\right]}{\left[\lim_{n\to\infty} 1\right]+\left[\lim_{n\to\infty}\frac{1}{n^3}\right]}$$

$$= \frac{0-3}{1+0} = \frac{-3}{1} = \boxed{-3}$$

(b) $$\lim_{n\to\infty} \frac{n}{n^2+1} = \frac{\lim_{n\to\infty} n}{\lim_{n\to\infty}(n^2+1)} = \frac{+\infty}{+\infty}$$

However we can rewrite this limit as:

$$\lim_{n\to\infty} \frac{(n)\frac{1}{n^2}}{(n^2+1)\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^2}}$$

$$= \frac{\lim_{n\to\infty}\frac{1}{n}}{1+\lim_{n\to\infty}\frac{1}{n^2}} = \frac{0}{1+0} = \frac{0}{1} = \boxed{0}$$

Can you see a general rule?

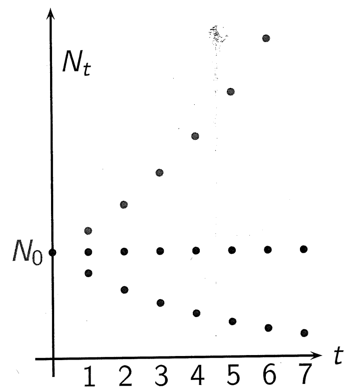
Example 4:

For $R > 0$, we know that exponential growth is given by

$$N_t = N_0 R^n \qquad n = 0, 1, 2, \ldots$$

The figure below indicates that

$$\lim_{n \longrightarrow \infty} N_t = \begin{cases} 0 & \text{if } 0 < R < 1 \\ N_0 & \text{if } R = 1 \\ \infty & \text{if } R > 1 \end{cases}$$



Example 5:

Find $\displaystyle\lim_{n \longrightarrow \infty} \frac{3 \cdot 4^n + 1}{4^n}$

$$\lim_{n \to \infty} \frac{3 \cdot 4^n + 1}{4^n} = \frac{\lim_{n \to \infty} (3 \cdot 4^n + 1)}{\lim_{n \to \infty} 4^n} = \frac{+\infty}{+\infty}$$

however we can rewrite the above limit as:

$$\lim_{n \to \infty} \left[\frac{3 \cdot 4^n}{4^n} + \frac{1}{4^n} \right] = \lim_{n \to \infty} \left[3 + \left(\frac{1}{4}\right)^n \right]$$

$$= \left[\lim_{n \to \infty} 3\right] + \left[\underbrace{\lim_{n \to \infty} \left[\frac{1}{4}\right]^n}_{0 \text{ as } R = \frac{1}{4}}\right] = 3 + 0$$

$$= \boxed{3}$$

Squeeze (Sandwich) Theorem for Sequences

Sometimes the limit of a sequence can be difficult to calculate and we need to employ some other techniques. One of those techniques is to use the Squeeze (Sandwich) Theorem for Sequences.

Squeeze (Sandwich) Theorem for Sequences

Consider three sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ and suppose there exists an integer N such that

$$a_n \leq b_n \leq c_n \qquad \text{for all} \quad n > N.$$

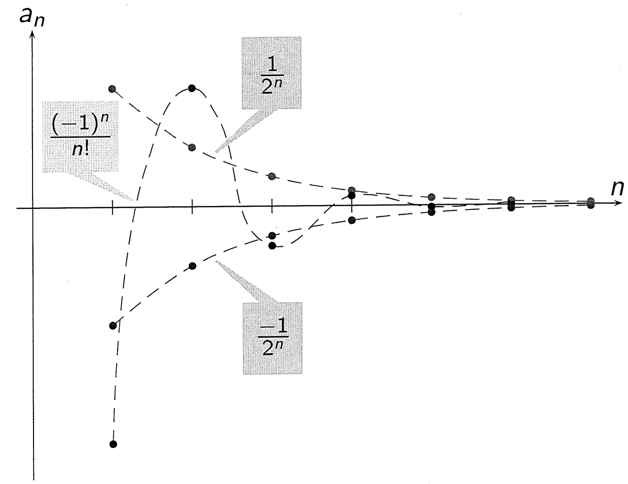
If $\lim\limits_{n\longrightarrow\infty} a_n = L = \lim\limits_{n\longrightarrow\infty} c_n$ then $\lim\limits_{n\longrightarrow\infty} b_n = L.$

The values in the following table and the graph on the left

n	$1/n!$	$1/2^n$
1	1	0.5
2	0.5	0.25
3	$0.1\overline{6}$	0.125
4	$0.041\overline{6}$	0.0625
5	$0.008\overline{3}$	0.03125
6	$0.0013\overline{8}$	0.015625
7	0.000198	0.0078125
$\vdots$	$\vdots$	$\vdots$

suggest that for $n \geq 4$ we have

$$\frac{-1}{2^n} \leq \frac{(-1)^n}{n!} \leq \frac{1}{2^n} \qquad n \geq 4.$$



So by the Squeeze Theorem it follows that

$$\lim_{n\longrightarrow\infty} (-1)^n \frac{1}{n!} = 0.$$

Example 6:

Find $\lim\limits_{n\longrightarrow\infty} \dfrac{2n+(-1)^n}{n}$

$$b_n = \frac{2n + (-1)^n}{n} = 2 + \frac{(-1)^n}{n}$$

Observe that $-1 \leq (-1)^n \leq 1$ for every n

Thus

$$\boxed{a_n = 2 - \frac{1}{n}} \leq \underbrace{2 + \frac{(-1)^n}{n}}_{b_n} \leq \boxed{2 + \frac{1}{n} = c_n}$$

and $\lim\limits_{n\to\infty}\left[2 - \frac{1}{n}\right] = 2 = \lim\limits_{n\to\infty}\left[2 + \frac{1}{n}\right]$

so that $\boxed{\lim\limits_{n\to\infty} \frac{2n+(-1)^n}{n} = 2}$

Example 7:

Find $\lim\limits_{n\longrightarrow\infty} \frac{5^n}{n!}$

Observe that

$$0 \le \frac{5^n}{n!} = \frac{\overbrace{5\cdot 5\cdot 5\cdot 5 \cdots 5\cdot 5}^{n \text{ times}}}{n\,(n-1)(n-2)\cdots 3\cdot 2\cdot 1}$$

we can regroup those terms as

$$\underbrace{\left[\frac{5}{n}\cdot\frac{5}{n-1}\cdot\frac{5}{n-2}\cdots\frac{5}{6}\right]}\cdot\frac{5}{5}\cdot\frac{5}{4}\cdot\frac{5}{3}\cdot\frac{5}{2}\cdot 5$$

$$\le \left(\frac{5}{6}\right)^{n-5}\cdot\frac{625}{24}$$

In other words: $0 \le \frac{5^n}{n!} \le \left(\frac{5}{6}\right)^{n-5}\cdot\frac{625}{24}$

But $\lim\limits_{n\to 0} 0 = 0 = \lim\limits_{n\to\infty}\left(\frac{5}{6}\right)^{n-5}\cdot\frac{625}{24}$ $\boxed{\text{as } 5/6 < 1}$

So $\boxed{\lim\limits_{n\to\infty}\frac{5^n}{n!} = 0}$