MA137 – Calculus 1 with Life Science Applications
Discrete-Time Models
Sequences and Difference Equations: **Limits**
(Section 2.2)

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Limits of Recursive Sequences

We now discuss how to find the limit when a_n is defined by a recursive sequence of the first order

$$a_{n+1} = f(a_n)$$

Finding an explicit expression for a_n is often not a feasible strategy, because solving recursions can be very difficult or even impossible.

How, then, can we say anything about the limiting behavior of a recursively defined sequence?

The following procedure will allow us to identify **candidates** for limits.

Fixed Points (or Equilibria)

Definition

A **fixed point** (or **equilibrium**) of a recursive sequence

$$a_{n+1} = f(a_n)$$

is a number $\widehat{a}$ that is left unchanged by the (updating function) g, that is,

$$\widehat{a} = f(\widehat{a})$$

Remark:
A fixed point is only a candidate for a limit; a sequence does not have to converge to a given fixed point (unless a_0 is already equal to the fixed point).

Example 1:

Let $a_{n+1} = 1 + \frac{1}{a_n}$. Find the fixed points of this recursion, and investigate the limiting behavior of a_n when $a_1 = 1$.

Consider the recursive sequence $a_{n+1} = 1 + \frac{1}{a_n}$

(Notice that $a_{n+1} = f(a_n)$ where $f(x) = 1 + \frac{1}{x}$)

To find the fixed points we need to solve for a in:

$$a = 1 + \frac{1}{a}$$

Multiply both sides by a : $a^2 = a\left(1 + \frac{1}{a}\right)$

$\iff a^2 = a + 1 \iff a^2 - a - 1 = 0$

and use now the quadratic formula:

$$a_{1,2} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2} < \begin{matrix} \frac{1+\sqrt{5}}{2} \\ \frac{1-\sqrt{5}}{2} \end{matrix}$$

Thus there are two fixed points:

$\hat{a}_1 = \frac{1+\sqrt{5}}{2} \cong 1.618$ GOLDEN RATIO $\qquad \hat{a}_2 = \frac{1-\sqrt{5}}{2} \cong -0.618$

Let's investigate $\lim_{n\to\infty} a_n$.

We already worked out a few terms of this sequence in an earlier lecture:

$a_{n+1} = 1 + \frac{1}{a_n}$

$a_1 = 1$

$a_2 = 1 + \frac{1}{a_1} = 1 + 1 = 2$

$a_3 = 1 + \frac{1}{a_2} = 1 + \frac{1}{2} = 3/2 = 1.5$

$a_4 = 1 + \frac{1}{a_3} = 1 + \frac{1}{3/2} = 1 + \frac{2}{3} = \frac{5}{3} = 1.6\bar{7}$

$a_5 = 1 + \frac{1}{a_4} = 1 + \frac{1}{5/3} = 1 + \frac{3}{5} = 8/5 = 1.6$

$a_6 = 1 + 1/a_5 = 1 + \frac{1}{8/5} = 1 + \frac{5}{8} = \frac{13}{8} = 1.625$

We realize that the n-th term of the sequence a_n is the quotient of two consecutive Fibonacci's numbers $(1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots)$

From the first few terms of the sequence we have worked out $a_1, a_2, \ldots, a_6$ it seems obvious that $\lim_{n\to\infty} a_n = \hat{a}_1 = \frac{1+\sqrt{5}}{2} \cong 1.618$

$\boxed{\text{Aside}}$, it takes quite some work and some mathematical skill to prove that there exists an explicit form of the Fibonacci's numbers

Namely: $$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

for $n = 1, 2, 3, \ldots\ldots$

Example 2:

Let $a_{n+1} = \sqrt{3a_n}$. Find the fixed points of this recursion, and investigate the limiting behavior of a_n when $a_0 = 1$.

$a_{n+1} = \sqrt{3a_n}$

(notice than $a_{n+1} = f(a_n)$ when $f(x) = \sqrt{3x}$)

To find the fixed points we have to solve

$$a = \sqrt{3a} \iff a^2 = (\sqrt{3a})^2$$ [i.e. we squared both sides]

$$\iff a^2 = 3a \iff a^2 - 3a = 0$$

$$\iff a(a-3) = 0 \iff \boxed{\hat{a}_1 = 0 \quad \hat{a}_2 = 3}$$

fixed points

We want to investigate $\lim_{n\to\infty} a_n$ with $a_0 = 1$

Then $a_0 = 1$; $a_1 = \sqrt{3a_0} = \sqrt{3} \cong 1.732$;

$a_2 = \sqrt{3a_1} \cong 2.279$; $a_3 = \sqrt{3a_2} \cong 2.615$

$a_4 = \sqrt{3a_3} \cong 2.8$; $a_5 = \sqrt{3a_4} \cong 2.898$

$a_6 = \sqrt{3a_5} \cong 2.949$; etc....

Hence all these calculations seem to suggest that $\boxed{\lim_{n\to\infty} a_n = 3}$

that is the limit is the fixed point $\hat{a}_2 = 3$.

Example 3:

Let $a_{n+1} = \dfrac{3}{a_n}$. Find the fixed points of this recursion, and investigate the limiting behavior of a_n when a_0 is not equal to a fixed point.

$a_{n+1} = \dfrac{3}{a_n}$

[that is $a_{n+1} = f(a_n)$ with $f(x) = \dfrac{3}{x}$]

Fixed points: we need to solve the equation

$$a = \frac{3}{a} \iff a^2 = 3 \iff a = \pm\sqrt{3}$$

Thus there are two fixed points: $\boxed{\hat{a}_1 = \sqrt{3}\ ;\ \hat{a}_2 = -\sqrt{3}}$

(1) Suppose that $a_0 = \sqrt{3} \Rightarrow a_1 = \dfrac{3}{a_0} = \dfrac{3}{\sqrt{3}} = \sqrt{3}$

$a_2 = \dfrac{3}{a_1} = \dfrac{3}{\sqrt{3}} = \sqrt{3} \Rightarrow$ hence $a_n = \sqrt{3}$ for all n.

(2) Similarly if we start with $a_0 = -\sqrt{3}$ we get that $a_1 = \dfrac{3}{a_0} = \dfrac{3}{-\sqrt{3}} = -\sqrt{3}$; $a_2 = \dfrac{3}{a_1} = \dfrac{3}{-\sqrt{3}} = -\sqrt{3}$

i.e. $a_n = -\sqrt{3}$ for all n.

(3) However, let's start for example with $a_0 = 2$. We have $a_1 = \dfrac{3}{a_0} = \dfrac{3}{2} = 1.5$

$a_2 = \dfrac{3}{a_1} = \dfrac{3}{3/2} = 2$; $a_3 = \dfrac{3}{a_2} = 3/2$; ....

Hence we conclude that even if we started close to the fixed point $\boxed{\hat{a}_1 = \sqrt{3}}$, i.e. we picked $a_0 = 2$. We got

$$a_0 = a_2 = a_4 = a_6 = a_8 = \cdots = 2$$
$$a_1 = a_3 = a_5 = a_7 = a_9 = \cdots = 3/2$$

Hence $\lim_{n\to\infty} a_n =$ does not exist

Comments

The previous examples illustrate that fixed points are only candidates for limits and that, depending on the initial condition, the sequence $\{a_n\}$ may or may not converge to a given fixed point.

If we know, however, that a sequence $\{a_n\}$ does converge, then the limit of the sequence must be one of the fixed points.

For this reason we say that a fixed point (or equilibrium) is **stable** if sequences that begin close to the fixed point approach that fixed point. It is called **unstable** if sequences that start close to the equilibrium move away from it.

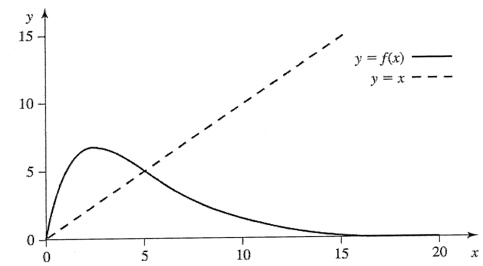
We will return to the relationship between fixed points and limits in Section 5.7, where we will learn methods that allow us to determine whether a sequence converges to a particular fixed point.

A Graphical Way to Find Fixed Points

There is a graphical method for finding fixed points, which we mention briefly below.

Given a recursion of the form $a_{n+1} = f(a_n)$, then we know that a fixed point $\hat{a}$ satisfies $\hat{a} = f(\hat{a})$.

This suggests that if we graph $y = f(x)$ and $y = x$ in the same coordinate system, then fixed points are located where the two graphs intersect, as shown in the picture below



Example 4:

(a) Consider the sequence recursively defined by the relation

$$a_{n+1} = 2a_n(1 - a_n) \qquad a_0 = 0$$

and assume that $\lim_{n\to\infty} a_n$ exists.
Find all fixed points of $\{a_n\}$, and use a table or other reasoning to guess which fixed point is the limiting value for the given initial condition.

(b) Same as in **(a)** but with $a_0 = 0.1$.

Notice that $a_{n+1} = 2a_n(1-a_n)$ is of the form

$a_{n+1} = f(a_n)$ where $f(x) = 2x(1-x)$
this is a parabola with downward concavity

To find the fixed points we need to solve

$a = 2a(1-a) \iff a = 0$ or

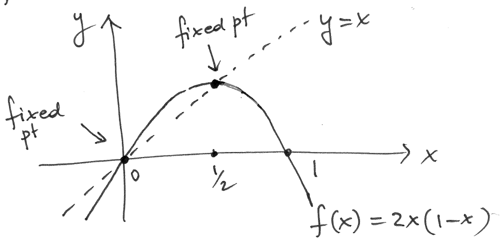
$1 = 2(1-a) \iff \frac{1}{2} = 1 - a \iff a = 1 - \frac{1}{2} = \frac{1}{2}$

Thus the fixed points are:

$\boxed{\hat{a}_1 = 0}$ or $\boxed{\hat{a}_2 = \frac{1}{2}}$

Notice the the fixed points are geometrically given by the intersection points between

$$y = f(x) = 2x(1-x) \quad \text{and} \quad y = x$$



About : $\lim_{n\to\infty} a_n$

(1) if $a_0 = 0$; $a_1 = 2a_0(1-a_0) = 0$;
$a_2 = 2a_1(1-a_1) = 0$ etc....

so $\boxed{\lim_{n\to\infty} a_n = 0}$

(2) Let's consider the case $a_0 = 0.1$
That is we start from a point that is very close to the equilibrium/fixed point 0.

$a_0 = 0.1$

$a_1 = 2a_0(1-a_0) = 2\cdot(0.1)\cdot(0.9) = 0.18$

$a_2 = 2a_1(1-a_1) = 2(0.18)(0.82) = 0.2952$

$a_3 = 2a_2(1-a_2) = 2(0.2952)(0.7048) = 0.4161$

$a_4 = \dots = 0.486$

Hence these values suggest $\boxed{\lim_{n\to\infty} a_n = 0.5}$

despite the fact that we started very close to 0.

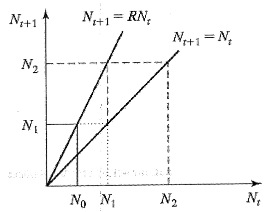
Appendix: Cobweb Plotter — Geogebra

Cobwebbing for $N_{t+1} = RN_t$

We can determine **graphically** whether a fixed point is stable or unstable.

The fixed points of exponential growth recursive sequence are found graphically where the graphs of $N_{t+1} = RN_t$ and $N_{t+1} = N_t$ intersect.

We see that the two graphs intersect where $N_t = 0$ only when $R \neq 1$.



We can use the two graphs on the left to follow successive population sizes. Start at N_0 on the horizontal axis. Since $N_1 = RN_0$, we find N_1 on the vertical axis, as shown by the solid vertical and horizontal line segments. Using the line $N_{t+1} = N_t$, we can locate N_1 on the horizontal axis by the dotted horizontal and vertical line segments.

Using the line $N_{t+1} = RN_t$ again, we can find N_2 on the vertical axis, as shown in the figure by the broken horizontal and vertical line segments. Using the line $N_{t+1} = N_t$ once more, we can locate N_2 on the horizontal axis and then repeat the preceding steps to find N_3 on the vertical axis, and so on.

This procedure is called **cobwebbing**.

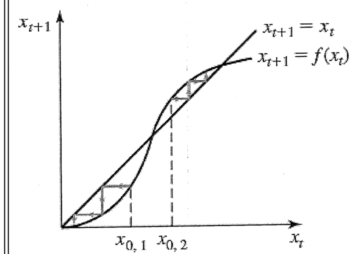
General Case

The general form of a first-order recursion is

$$x_{t+1} = f(x_t), \qquad t = 0, 1, 2, \ldots$$

- To find fixed points **algebraically**, we solve $x = f(x)$.
- To find them **graphically**, we look for points of intersection of the graphs of $x_{t+1} = f(x_t)$ and $x_{t+1} = x_t$.

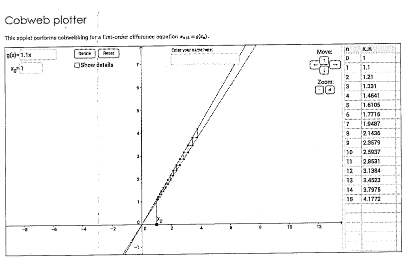
The graphs in the picture intersect more than once, which means that there are multiple equilibria. We can use the cobwebbing procedure from the previous page to graphically investigate the behavior of the difference equation for different initial values.



Two cases are shown in the picture, one starting at $x_{0,1}$ and the other at $x_{0,2}$. We see that x_t converges to different values, depending on the initial value.

Example 5

The recursive sequence $x_{n+1} = Rx_n$ has only one fixed point: $\widehat{x} = 0$.

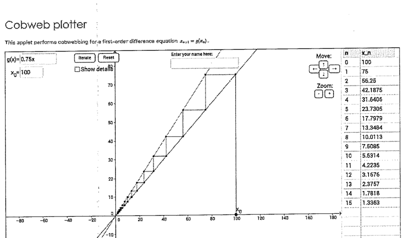


The recursive sequence

$$x_0 = 1 \qquad x_{n+1} = 1.1x_t$$

does not converges to $\widehat{x} = 0$.

$$\lim_{t\to\infty} x_t = \infty \quad \text{(or DNE)}$$



The recursive sequence

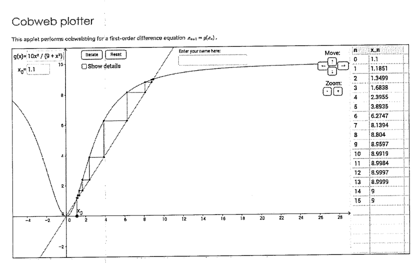
$$x_0 = 100 \qquad x_{n+1} = 0.75x_t$$

converges to $\widehat{x} = 0$.

$$\lim_{t\to\infty} x_t = 0$$

Example 6

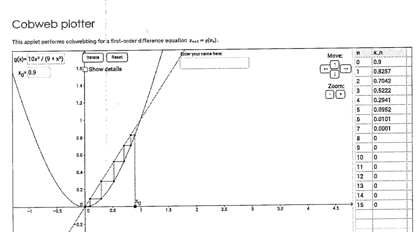
One can easily check that the recursive sequence $x_{n+1} = \dfrac{10x_t^2}{9 + x_t^2}$ has the following three fixes points: $\widehat{x} = 0, 1, 9$.



The recursive sequence

$$x_0 = 1.1 \qquad x_{n+1} = \frac{10x_t^2}{9 + x_t^2}$$

converges to the fixes point $\widehat{x} = 9$.



The recursive sequence

$$x_0 = 0.9 \qquad x_{n+1} = \frac{10x_t^2}{9 + x_t^2}$$

converges to the fixes point $\widehat{x} = 0$.