MA 137 – Calculus 1 with Life Science Applications
Limits
(Section 3.1)

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Computing a limit means computing what happens to the value of a function as the variable in the expression gets closer and closer to (but does not equal) a particular value.

Intuitive Definition

Let f be a function of x. The expression $\lim_{x\to c} f(x) = L$ means that as x gets closer and closer to c, through values both smaller and larger than c, but not equal to c, then the values of $f(x)$ get closer and closer to the value L.

Note 1: If $\lim_{x\to c} f(x) = L$ and L is a finite number, we say that the limit exists and that $f(x)$ **converges** to L as x tends to c.
If the limit does not exist, we say that $f(x)$ **diverges** as x tends to c.

Note 2: when finding the limit of $f(x)$ as x approaches c, we do not simply plug c into $f(x)$. (OK...often we do!)
In fact, we will see examples in which $f(x)$ is not even defined at $x = c$.
The value of $f(c)$ is irrelevant when we compute the value of $\lim_{x\to c} f(x)$.

Example 1:

Compute $\lim_{x\to 2} \dfrac{x^2+8}{x+2}$.

x gets close to 2 from the left				
x	1.8	1.9	1.99	1.999
$f(x) = \dfrac{x^2+8}{x+2}$				

x gets close to 2 from the right				
2.001	2.01	2.1	2.2	x
				$f(x) = \dfrac{x^2+8}{x+2}$

Using a calculator or even better an Excel spreadsheet we have that

x	1.8	1.9	1.99	1.999
$f(x)$	2.9579	2.9769	2.9975	2.9998

for $f(x) = \dfrac{x^2+8}{x+2}$

and

x	2.001	2.01	2.1	2.2
$f(x)$	3.0003	3.0025	3.0268	3.0571

so it seems that the values of $f(x)$ approach 3 as x approaches 2: $\lim_{x\to 2} \dfrac{x^2+8}{x+2} = \boxed{3}$

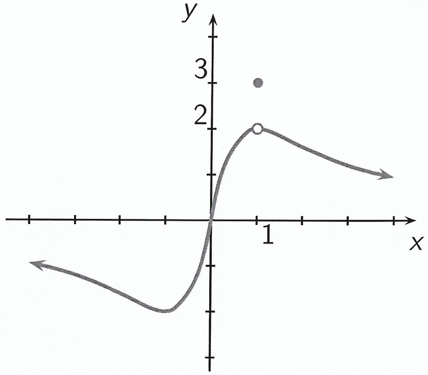
Note that in this case: $f(2) = \dfrac{2^2+8}{2+2} = \dfrac{12}{4} = \boxed{3}$

Example 2:

The graph of the function

$$g(x) = \begin{cases} \dfrac{4x}{x^2+1} & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$$

is shown to the right.



Compute $\displaystyle\lim_{x\to 1} g(x)$.

x	0.8	0.9	0.99	1.001	1.1	1.2
$g(x)$	1.95121	1.98895	1.9999	1.9999	1.99095	1.96721

$$g(x) = \begin{cases} \dfrac{4x}{x^2+1} & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$$

notice that from the table of values:

x	0.8	0.9	0.99	1.001	1.1	1.2
$g(x)$	1.95121	1.98895	1.9999	1.9999	1.99095	1.96721

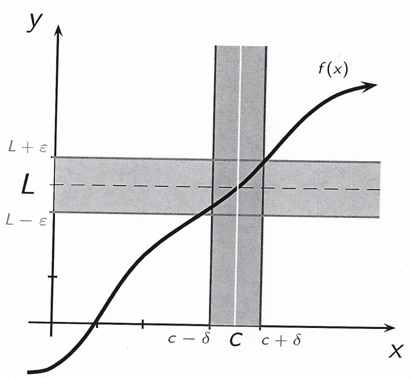
it seems that the values of $g(x)$ tend to 2 as x approaches 1:

$\boxed{\displaystyle\lim_{x\to 1} g(x) = 2}$. Note $\boxed{g(1) = 3}$

so $\boxed{\displaystyle\lim_{x\to 1} g(x) \neq g(1)}$

Formal Definition

The statement $\displaystyle\lim_{x\to c} f(x) = L$ means that, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$.



The formal definition says that no matter how small we choose the horizontal strip about $y=L$, we can always choose a small strip about $x=c$ so that

whenever $c-\delta < x < c+\delta$ $\quad x \neq c$

then $L-\varepsilon < f(x) < L+\varepsilon$

i.e. if the values of x is sufficiently close to c then the corresponding value $f(x)$ is close to L.

Limit Laws

The operations of arithmetic, namely, addition, subtraction, multiplication, and division, all behave reasonably with respect to the idea of getting closer to as long as nothing illegal happens.

This is summarized by the following laws:

If $\lim\limits_{x \longrightarrow c} f(x)$ and $\lim\limits_{x \longrightarrow c} g(x)$ exist and a is a constant, then

1. $\lim\limits_{x \longrightarrow c} [f(x) + g(x)] = [\lim\limits_{x \longrightarrow c} f(x)] + [\lim\limits_{x \longrightarrow c} g(x)]$
2. $\lim\limits_{x \longrightarrow c} [a\, f(x)] = a [\lim\limits_{x \longrightarrow c} f(x)]$
3. $\lim\limits_{x \longrightarrow c} [f(x) \cdot g(x)] = [\lim\limits_{x \longrightarrow c} f(x)] \cdot [\lim\limits_{x \longrightarrow c} g(x)]$
4. $\lim\limits_{x \longrightarrow c} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \longrightarrow c} f(x)}{\lim\limits_{x \longrightarrow c} g(x)}$, provided $\lim\limits_{x \longrightarrow c} g(x) \neq 0$

Theorem (Substitution Theorem 1)

If $p(x)$ is a polynomial, then $\lim\limits_{x \longrightarrow c} p(x) = p(c)$.

Proof: A polynomial is a sum of terms, say $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$.

The result now follows from the Limit Laws:

$$\lim_{x \to c} p(x) = \lim_{x \to c} \left[a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \right]$$

(the limit of the sum is the sum of the limits)

$$= \lim_{x \to c} [a_n x^n] + \lim_{x \to c} [a_{n-1} x^{n-1}] + \cdots + \lim_{x \to c} [a_2 x^2] + \lim_{x \to c} [a_1 x] + \lim_{x \to c} [a_0]$$

(each of the terms is a product and the limit of the product is the product of the limits)

$$= \lim_{x \to c} [a_n] \lim_{x \to c} [x^n] + \lim_{x \to c} [a_{n-1}] \lim_{x \to c} [x^{n-1}] + \cdots + \lim_{x \to c} [a_2] \lim_{x \to c} [x^2] + \lim_{x \to c} [a_1] \lim_{x \to c} [x] + \lim_{x \to c} [a_0]$$

(each of these terms is either a constant or a power of x)

$$= a_n c^n + a_{n-1} c^{n-1} + \cdots + a_2 c^2 + a_1 c + a_0 = p(c)$$

Theorem (Substitution Theorem 2)

If $f(x)$ is a rational function, that is $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, then

$$\lim_{x \longrightarrow c} f(x) = \lim_{x \to c} \frac{p(x)}{q(x)} = \frac{\lim\limits_{x \to c} p(x)}{\lim\limits_{x \to c} q(x)} = \frac{p(c)}{q(c)} = f(c),$$

provided $q(c) \neq 0$.

The usual issue is that we often have to compute limits when these conditions are not met.

Example 3:

(a) Compute $\lim\limits_{x \to 1} \dfrac{x^2 - 2x + 1}{x + 1}$.

(b) Suppose $\lim\limits_{x \to 3} f(x) = -2$ and $\lim\limits_{x \to 3} g(x) = 4$. Determine

$$\lim_{x \to 3} \left[(x+1) \cdot f(x)^2 + \frac{x+2}{g(x)} \right]$$

(a) $$\lim_{x\to 1}\frac{x^2-2x+1}{x+1} = \frac{\lim_{x\to 1}(x^2-2x+1)}{\lim_{x\to 1}(x+1)} =$$

$$= \frac{\lim_{x\to 1}(x^2) + \lim_{x\to 1}(-2x) + \lim_{x\to 1} 1}{\left(\lim_{x\to 1} x\right) + \lim_{x\to 1} 1} =$$

$$= \frac{\left[\lim_{x\to 1} x\right]^2 - 2\left(\lim_{x\to 1} x\right) + 1}{1+1}$$

$$= \frac{1^2 - 2(1) + 1}{2} = \frac{0}{2} = \boxed{0}$$

this is Substitution Thm 1

(b) $$\lim_{x\to 3}\left[(x+1)f(x)^2 + \frac{x+2}{g(x)}\right] =$$

$$= \lim_{x\to 3}\left[(x+1)f(x)^2\right] + \lim_{x\to 3}\frac{x+2}{g(x)}$$

$$= \left[\lim_{x\to 3}(x+1)\right]\left[\lim_{x\to 3} f(x)^2\right] + \frac{\lim_{x\to 3}(x+2)}{\lim_{x\to 3} g(x)}$$

$$= \left[\lim_{x\to 3} x + \lim_{x\to 3} 1\right]\left[\underbrace{\lim_{x\to 3} f(x)}_{=-2}\right]^2 + \frac{\left[\lim_{x\to 3} x\right] + \lim_{x\to 3} 2}{\underbrace{\lim_{x\to 3} g(x)}_{=4}}$$

$$= (3+1)(-2)^2 + \frac{3+2}{4} = 16 + \frac{5}{4} = \frac{69}{4} \cong 17.25$$

When Limits Fail to Exist

There are two basic ways that a limit can fail to exist.

(a) The function attempts to approach multiple values as $x \to c$.

Geometrically, this behavior can be seen as a jump in the graph of a function.

Algebraically, this behavior typically arises with piecewise defined functions.

(b) The function grows without bound as $x \to c$.

Geometrically, this behavior can be seen as a vertical asymptote in the graph of a function.

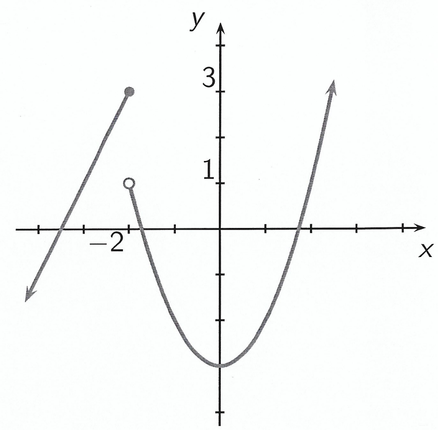
Algebraically, this behavior typically arises when the denominator of a function approaches zero.

Example 4:

The graph of the function

$$h(x) = \begin{cases} x^2 - 3 & \text{if } x > -2 \\ 2x + 7 & \text{if } x \le -2 \end{cases}$$

is shown to the right.



Analyze $\lim_{x\to -2} h(x)$.

If we approach -2 from the left, the function $h(x)$ is defined by $2x+7$ hence

$$\lim_{\substack{x\to -2 \\ \text{from the left}}} h(x) = \lim_{\substack{x\to -2 \\ \text{from the left}}} (2x+7) = -4+7 = \boxed{3}$$

If we approach -2 from the right, the function $h(x)$ is defined by x^2-3 hence

$$\lim_{\substack{x\to -2 \\ \text{from the right}}} h(x) = \lim_{\substack{x\to -2 \\ \text{from the right}}} (x^2-3) = (-2)^2-3 = \boxed{1}$$

Since the values do not coincide $\lim_{x\to -2} h(x)$ DOES NOT EXIST!

One-sided Limits

The previous example brings us to the following notions:

One-sided limits

A one-sided limit expresses what happens to the values of an expression as the variable in the expression gets closer and closer to some particular value c from either the left on the number line (that is, through values less than c) or from the right on the number line (that is, through values greater than c).
The notation is:

$$\underbrace{\lim_{x\to c^-} f(x)}_{\text{limit from the left of } c} \qquad \underbrace{\lim_{x\to c^+} f(x)}_{\text{limit from the right of } c}$$

Fact: $\lim_{x\to c} f(x)$ exists if and only if both $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist and have the same value.

Example 5:

(a) Analyze $\lim_{x\to 1} \dfrac{5}{(x-1)^2}$.

(b) Analyze $\lim_{x\to 1} \dfrac{2}{x-1}$.

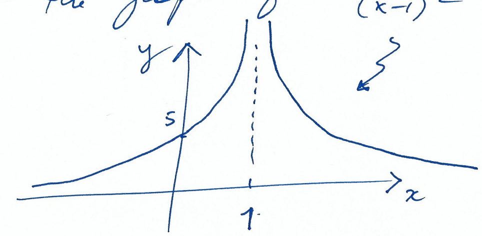
(c) Analyze the limit $\lim_{x\to 0} \dfrac{2}{\sqrt{x}}$.

(a) $\lim_{x\to 1} \dfrac{5}{(x-1)^2}$ if we build a table of values nearby $x=1$ we obtain

x	0.9	0.99 ...	... 1.01	1.1
$\dfrac{5}{(x-1)^2}$	500	50,000 ...	... 50,000	50

it seems that $\lim_{x\to 1} \dfrac{5}{(x-1)^2} = +\infty$ i.e. D.N.E.

look at the graph of $\dfrac{5}{(x-1)^2}$



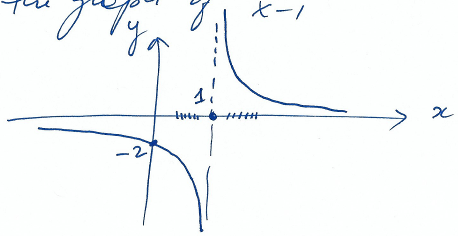
(b) $\lim_{x\to 1} \frac{2}{x-1}$ if we build a table of values nearby $x=1$ we obtain

x	0.9	0.99	$0.999 \cdots$	$\cdots\ 1.001$	1.01	1.1
$\frac{2}{x-1}$	-20	-200	-2000	2000	200	20

hence $\lim_{x\to 1^-} \frac{2}{x-1} = -\infty$ $\lim_{x\to 1^+} \frac{2}{x-1} = +\infty$

no matter what $\lim_{x\to 1} \frac{2}{x-1}$ D.N.E

If you look at the graph of $\frac{2}{x-1}$ we have



(c) Analyze $\lim_{x\to 0} \frac{2}{\sqrt{x}}$.

First, the limit means $\lim_{x\to 0^+} \frac{2}{\sqrt{x}}$ as $\sqrt{x}$ is not defined for negative values of x.

Build a table of values:

x	$\cdots 0.001$	0.01	0.1
$2/\sqrt{x}$	$\cdots 63.25$	20	6.325

it seems $\lim_{x\to 0^+} \frac{2}{\sqrt{x}} = +\infty$ or D.N.E.

Try to graph $2/\sqrt{x}$ and see how the graph looks like near 0^+.

The most interesting and important situation with limits is when a substitution yields 0/0.
The result 0/0 yields absolutely no information about the limit. It does not even tell us that the limit does not exist. The only thing it tells us is that we have to do more work to determine the limit.

Example 6:

Find the limit $\lim_{x\to 3} \frac{x^2-2x-3}{x-3}$.

$$\lim_{x\to 3} \frac{x^2-2x-3}{x-3} \underset{\text{substitution Thm 2}}{=} \frac{\lim_{x\to 3} x^2-2x-3}{\lim_{x\to 3} x-3} = \frac{3^2-2(3)-3}{3-3} = \frac{0}{0}$$

What now?

$$\lim_{x\to 3} \frac{x^2-2x-3}{x-3} = \lim_{x\to 3} \frac{\cancel{(x-3)}(x+1)}{\cancel{(x-3)}}$$

$$= \lim_{x\to 3} (x+1) = 3+1 = \boxed{4}$$

by substitution Thm I.

Example 7: (Online Homework HW07, #5)

Guess the value of the limit (if it exists) by evaluating the function at values close to where the limit is to be done.

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$$

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \frac{0}{0}$$

To make an educated guess, let's build a table of values:

x	-0.1	-0.01	-0.001	-0.0001 -->	<-- 0.0001	0.001	0.01	0.1
$\frac{e^x-1-x}{x^2}$	0.48374	0.49834	0.49983	0.49998 -->	<- 0.50002	0.50017	0.50167	0.51709

it seems that $\boxed{\lim_{x\to 0} \frac{e^x-1-x}{x^2} = 0.5}$

Wow!!!

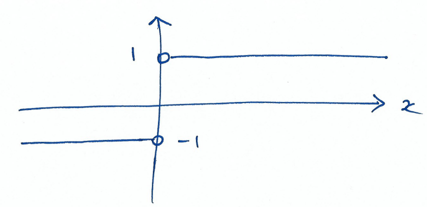
We will try to justify this answer... soon.

Example 8:

Find the limits

$$\lim_{x\to 0^+} \frac{|x|}{x} \qquad \lim_{x\to 0^-} \frac{|x|}{x} \qquad \lim_{x\to 0} \frac{|x|}{x}.$$

Graph $\frac{|x|}{x}$! For $x>0$ $|x|=x$ so $\frac{|x|}{x} = \frac{x}{x} = 1$; For $x<0$ $|x|=-x$ so $\frac{|x|}{x} = \frac{-x}{x} = -1$. Thus



Function is not defined when $\underline{\underline{x=0}}$

$$\lim_{x\to 0^-} \frac{|x|}{x} = -1 \qquad \lim_{x\to 0^+} \frac{|x|}{x} = 1$$

hence $\lim_{x\to 0} \frac{|x|}{x} = \underline{\underline{\text{D.N.E.}}}$

Example 9: (Neuhauser, Example 9, p. 108)

Find the limit $\lim_{x \to 0} \frac{\sqrt{x^2+16}-4}{x^2}$

If we build a table of values around 0

x	0.1	0.01	0.001	0.0001	0.00001	0.000001
$f(x)$	0.12498	0.125	0.125	0.125	0.125	0.12434

0.0000001	0.00000001	....
0.17764	0	...

These # have been calculated with an Excel spreadsheet

So the calc. suggest $\lim_{x \to 0} \frac{\sqrt{x^2+16}-4}{x^2} = 0$?

HOWEVER even calculators may round up in the wrong way.....

$$\lim_{x \to 0} \frac{\sqrt{x^2+16}-4}{x^2} = \lim_{x \to 0} \frac{(\sqrt{x^2+16}-4)}{x^2} \cdot \frac{(\sqrt{x^2+16}+4)}{\sqrt{x^2+16}+4}$$

$$= \lim_{x \to 0} \frac{(\sqrt{x^2+16})^2 - 4^2}{x^2(\sqrt{x^2+16}+4)} =$$

$$= \lim_{x \to 0} \frac{x^2+16-16}{x^2(\sqrt{x^2+16}+4)} = \lim_{x \to 0} \frac{x^2}{x^2(\sqrt{x^2+16}+4)}$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{x^2+16}+4} = \frac{1}{\sqrt{0^2+16}+4} = \frac{1}{8}$$

$$= 0.125$$