MA 137 – Calculus 1 with Life Science Applications
Continuity
(Section 3.2)

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Intuitive Examples

(a) What is the main difference between the following two functions?

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x \neq 2 \\ 5 & x = 2 \end{cases} \qquad g(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x \neq 2 \\ 4 & x = 2 \end{cases}$$

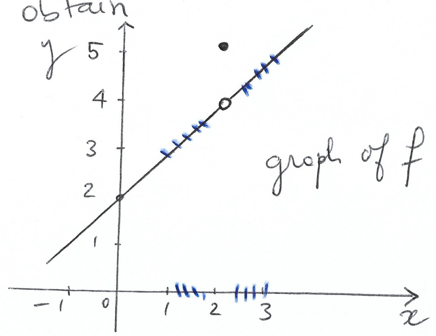
How does this difference translate when we graph f and g?

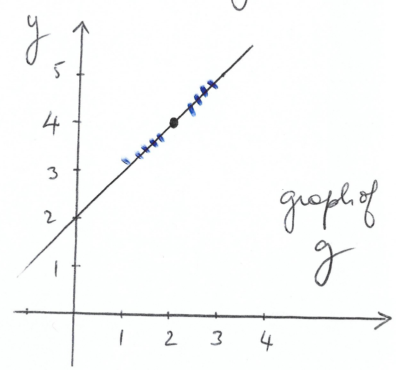
(b) What is the main difference between the following two functions?

$$\widetilde{f}(x) = \begin{cases} \dfrac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \qquad \widetilde{g}(x) = \begin{cases} \dfrac{|x|}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

How does this difference translate when we graph $\widetilde{f}$ and $\widetilde{g}$?

(a) If we plot the functions f and g we obtain

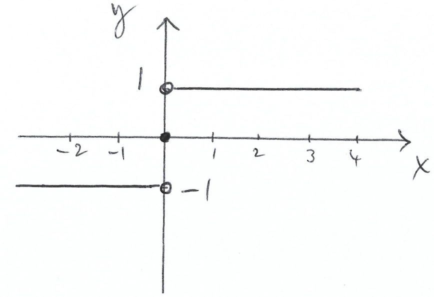




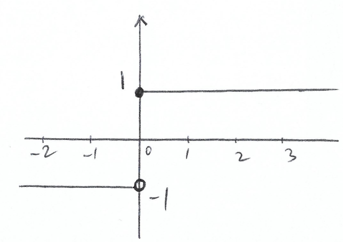
this is because you should realize that $\boxed{\dfrac{x^2-4}{x-2}} = \dfrac{(\cancel{x-2})(x+2)}{(\cancel{x-2})} = \boxed{x+2}$ (for $x \neq 2$)

(b)

graph of $\widetilde{f}$



graph of $\widetilde{g}$



$$\widetilde{f} = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \qquad \widetilde{g} = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

The previous example, (**a**), suggests that the definition of g at $x = 2$ is the one that fills the "hole" in the graph!!

Unfortunately, in the second case, (**b**), there is no way that we can assign a value to either $\widetilde{f}$ or $\widetilde{g}$ such that the graphs have no jump at $x = 0$.

Intuitively, we define a function to be continuous if "we can draw its graph without lifting our pencil from the paper."

In other words, this means that there are "no holes" in the graph.

In the first case, (**a**), the discontinuity of f at $x = 2$ could be removed and we got g that is continuous at $x = 2$. In the other case, (**b**), the discontinuity at $x = 0$ is not removable.

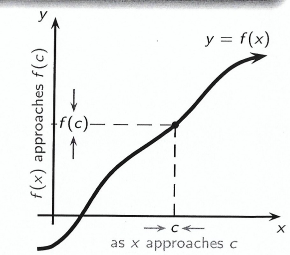
Continuity at a Point

Formal Definition

A function f is **continuous at a point** $x = c$ if $\lim_{x \to c} f(x) = f(c)$.

To check whether f is continuous at $x = c$, we need to check the following conditions:

1. $f(x)$ is defined at $x = c$;
2. $\lim_{x \to c} f(x)$ exists;
3. $\lim_{x \to c} f(x)$ is equal to $f(c)$.



Thus a continuous function f has the property that a small change in x produces only a small change in $f(x)$. In fact, the change in $f(x)$ can be kept as small as we please by keeping the change in x sufficiently small.

If any of those three conditions fails, f is **discontinuous** at $x = c$.

Continuity on an Interval

- A function f is continuous from the **right** at a point $x = c$ if
$$\lim_{x \to c^+} f(x) = f(c)$$
and f is continuous from the **left** at a point $x = c$ if
$$\lim_{x \to c^-} f(x) = f(c)$$
- We say that a function f is continuous on an **interval** I if f is continuous for all $x \in I$.
- If I is a closed interval, then continuity at the left (and, respectively, right) endpoint of the interval means continuous from the right (and, respectively, left).
- **Geometrically**, you can think of a function that is continuous at every point in an interval as a function whose graph has no break in it. The graph can be drawn without removing your pencil from the paper.

Example 1:

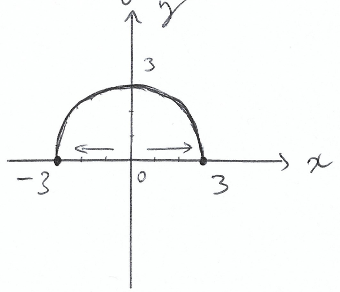
Make a graph of the function $f(x) = \sqrt{9 - x^2}$.

For which values of x is the function f continuous?

the graph of $f(x) = \sqrt{9-x^2}$ corresponds to half of the circle of radius 3 centered at the origin

This is because $y = \sqrt{9-x^2} \geq 0$

$\iff y^2 = 9 - x^2 \iff \boxed{x^2+y^2=9}$



the function is continuous for all $-3 < x < 3$

at $x=-3$ it is right continous

at $x=3$ it is left continuous

A Helpful Rewrite and a Few Comments

Our definition says that if $f(x)$ is continuous at a point $x = c$ then we can take the limit inside the function f:

$$\lim_{x \to c} f(x) = f(\lim_{x \to c} x).$$

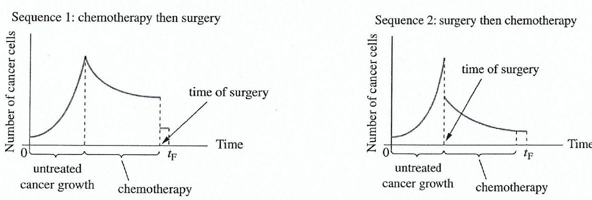
We like continuous functions is because they are...predictable.

- Assume you are watching a bird flying and then close your eyes for a second. When you open your eyes, you know that the bird will be somewhere around the location where you last saw it.
- As you move up a mountain, flora is a discontinuous function of altitude. There is the 'tree line,' below which the dominant plant species are pine and spruce and above which the dominant plant species are low growing brush and grasses.

Chemotherapy and surgery are two frequently used treatments for many cancers. How should they be used—chemotherapy first and then surgery, or the other way around? It is a non trivial matter.

For the case of ovarian cancer, researchers built mathematical models that track number of cancer cells as chemotherapy then surgery or surgery then chemotherapy.

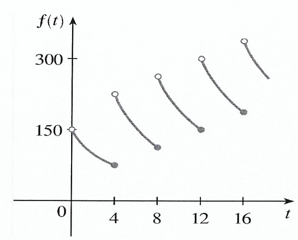
M. Kohandel et al.
Mathematical modeling of ovarian cancer treatments: Sequencing of surgery and chemotherapy
Journal of Theoretical Biology **242**, 62-68, 2006



Both functions have a discontinuity at moment of surgery (cancer cells removed). The size of the jump is highly relevant to decision about sequence of treatments applied.

Example 2:

A patient receives a 150-mg injection of a drug every 4 hours. The graph shows the amount $f(t)$ of the drug in the bloodstream after t hours.



At what values of t does $f(t)$ have discontinuities?
What type of discontinuities does $f(t)$ have?

The discontinuities of f are jumps at $x=4$, $x=8$, $x=12$, $x=16$ ... that is every time there is an injection of the drug.

Example 3:

Let $f(x) = \lfloor x \rfloor$ be the function that associates to any value of x the greatest integer less than or equal to x.

Make a graph of the function $y = \lfloor x \rfloor$.

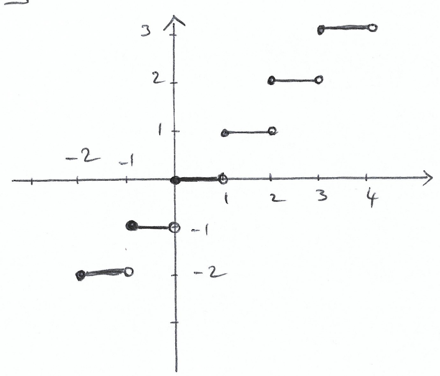
For which values of x is the function f continuous?

$f(x) = \lfloor x \rfloor =$ greatest integer less than or equal to x

So for example $\lfloor 1 \rfloor = 1$ $\lfloor 1.2 \rfloor = 1$

$\lfloor 1.99 \rfloor = 1$ but $\lfloor 2.1 \rfloor = 2$ $\lfloor 2.9 \rfloor = 2$

$\lfloor 3 \rfloor = 3$ etc...



the function is continuous whenever x is not an integer

Continuity and Operations on Functions

Using the limit laws, the following statements hold for combinations of continuous functions:

If α is a constant and the functions f and g are continuous at $x = c$, then the following functions are continuous at $x = c$:

1. $\alpha \cdot f$
2. $f \pm g$
3. $f \cdot g$
4. f/g provided that $g(c) \neq 0$

Proof of 2.: Since f and g are continuous at $x = c$, it follows that

$$\lim_{x \to c} f(x) = f(c) \quad \text{and} \quad \lim_{x \to c} g(x) = g(c).$$

We can apply one of the rules of Limit Laws and find that

$$\lim_{x \to c} (f \pm g)(x) \stackrel{\text{def.}}{=} \lim_{x \to c} [f(x) \pm g(x)] \stackrel{\text{rule}}{=} [\lim_{x \to c} f(x)] \pm [\lim_{x \to c} g(x)] \stackrel{\text{cont.}}{=} f(c) \pm g(c) \stackrel{\text{def.}}{=} (f \pm g)(c).$$

Example 4: (Online Homework HW08, # 12)

If f and g are continuous functions with

$$f(3) = 5 \quad \text{and} \quad \lim_{x\to 3}[2f(x) - g(x)] = 4,$$

find $g(3)$.

f and g are continuous with

$f(3)=5$ and $\lim_{x\to 3}\left[2f(x) - g(x)\right] = 4$

Notice that $2f - g$ is also a continuous function. So:

$$4 = 2f(3) - g(3) \overset{\text{continuity}}{=} \lim_{x\to 3}\left[2f(x) - g(x)\right]$$

$\Longrightarrow \quad 4 = 2\cdot 5 - g(3)$

So $\quad g(3) = 10 - 4 = 6$

Catalogue of Continuous Functions

Many of the elementary functions are continuous *wherever they are defined.* Here is a list:

1. Polynomial functions[a]
2. Rational functions[a]
3. Power functions
4. Trigonometric functions
5. Exponential functions of the form a^x, $a > 0$ and $a \neq 1$
6. Logarithmic functions of the form $\log_a x, a > 0$ and $a \neq 1$

[a] For polynomials and rational functions, this statement follows immediately from the fact that certain combinations of continuous functions are continuous.

The phrase "wherever they are defined" helps us to identify points where a function might be discontinuous. For instance, the function $1/(x-3)$ is defined only for $x \neq 3$, and the function $\sqrt{x-2}$ is defined only for $x \geq 2$.

Example 5: (Online Homework HW08, # 17)

Consider the function
$$f(x) = \begin{cases} b - 2x & \text{if } x < -4 \\[1ex] \dfrac{-96}{x-b} & \text{if } x \geq -4 \end{cases}$$

Find the two values of b for which f is a continuous function at $x = -4$.

Draw a graph of f.

$$f(x)=\begin{cases} b-2x & x<-4 \\ \dfrac{-96}{x-b} & x\geq -4 \end{cases}$$

We need to have $\lim\limits_{x\to -4^-} f(x) = \lim\limits_{x\to -4^+} f(x)$

I.e. $\lim\limits_{x\to -4^-} (b-2x) = \lim\limits_{x\to -4^+} \dfrac{-96}{x-b}$

$\iff b-2(-4) = \dfrac{-96}{-4-b}$

$\iff b+8 = \dfrac{96}{4+b} \iff (b+8)(4+b)=96$

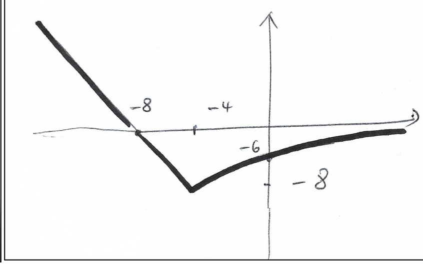
$\iff b^2+12b-64=0 \iff (b+16)(b-4)=0$

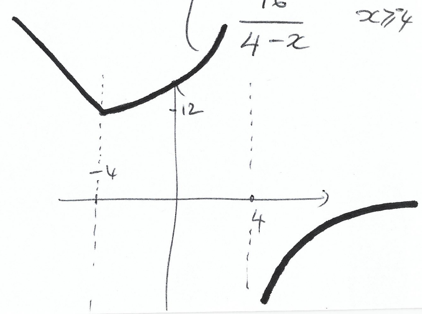
$\therefore$ $b=-16$ and $b=4$

We get 2 possibilities

$$f_1(x)=\begin{cases} -16-2x & x<-4 \\ \dfrac{-96}{x+16} & x\geq -4 \end{cases}$$

$$f_2(x)=\begin{cases} 4-2x & x<-4 \\ \dfrac{96}{4-x} & x\geq 4 \end{cases}$$





Continuity and Composition of Functions

Another way of combining continuous functions to get a new continuous function is to form their composition.

Theorem

If $g(x)$ is continuous at $x=c$ and $f(x)$ is continuous at $x=g(c)$, then $(f\circ g)(x)$ is continuous at $x=c$.
In particular,

$$\lim_{x\to c}(f\circ g)(x) = \lim_{x\to c} f[g(x)] = f[\lim_{x\to c} g(x)] = f(g(c)) = (f\circ g)(c)$$

In other words, the above theorem says that
"composition of continuous functions is continuous."

Example 6: (Online Homework HW08, # 10)

Use continuity to evaluate $\lim\limits_{x\to 1} e^{x^2-3x+5}$.

the exponential function and any polynomial function are continuous so e^{x^2-3x+5} is also continuous because it is the composition of 2 continuous functions:

$$\lim_{x\to 1}\left(e^{x^2-3x+5}\right) = e^{\lim_{x\to 1}(x^2-3x+5)}$$

$$= e^{1^2-3(1)+5} = \boxed{e^3}$$