MA 137 – Calculus 1 with Life Science Applications
The Sandwich Theorem and Some Trigonometric Limits
(Section 3.4)

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The Sandwich (Squeeze) Theorem

Suppose we want to calculate $\lim_{x\to\infty} e^{-x}\cos(10x)$.

We soon realize that none of the rules we have learned so far apply. Although $\lim_{x\to\infty} e^{-x} = 0$, we find that $\lim_{x\to\infty} \cos(10x)$ does not exist as the function $\cos(10x)$ oscillates between -1 and 1.

We need to employ some other techniques. One of these techniques is to use the Squeeze (Sandwich) Theorem.

Sandwich (Squeeze) Theorem

Consider three functions $f(x)$, $g(x)$ and $h(x)$ and suppose for all x in an open interval that contains c (except possibly at c) we have

$$f(x) \le g(x) \le h(x).$$

If $\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x)$ then $\lim_{x\to c} g(x) = L$.

From the inequality

$$-1 \le \cos(10x) \le 1$$

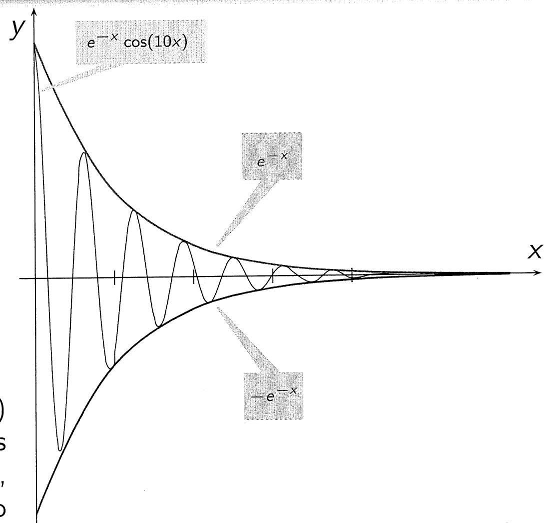
it follows that (as $e^{-x} > 0$, always)

$$-e^{-x} \le e^{-x}\cos(10x) \le e^{-x}$$

Then, since

$$\lim_{x\to\infty}(-e^{-x}) = 0 = \lim_{x\to\infty} e^{-x}$$

our function $g(x) = e^{-x}\cos(10x)$ is squeezed between the functions $f(x) = -e^{-x}$ and $h(x) = e^{-x}$, which both go to 0 as x tends to infinity.



So by the Squeeze Theorem it follows that

$$\lim_{x\to\infty} e^{-x}\cos(10x) = 0.$$

Example 1: (Online Homework HW10, # 2)

Suppose $-8x - 22 \le f(x) \le x^2 - 2x - 13.$

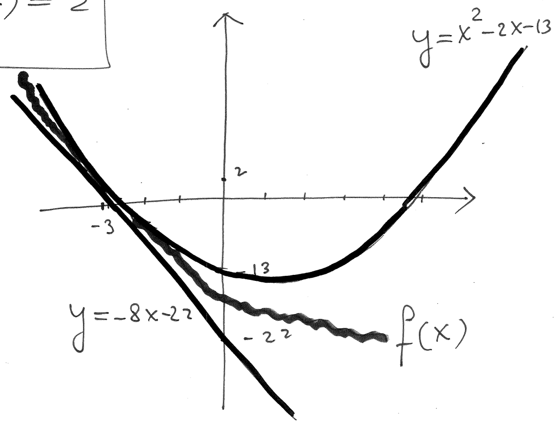
Use this to compute $\lim_{x\to -3} f(x)$.

Note that $\lim\limits_{x\to-3}(-8x-22) \underset{\text{substitution theorem}}{=} -8(-3)-22 = \boxed{2}$

Moreover $\lim\limits_{x\to-3}(x^2-2x-13) = (-3)^2-2(-3)-13 = \boxed{2}$

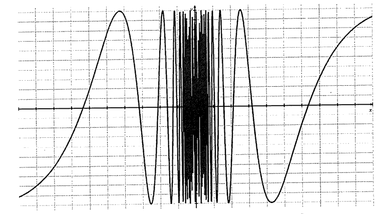
Hence by the sandwich (Squeeze) Theorem

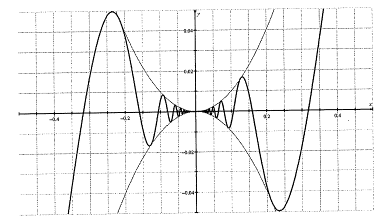
$$\boxed{\lim_{x\to-3} f(x) = 2}$$



Example 2: (Neuhauser, Example # 2, p. 127)

Find $\lim\limits_{x\to0} x^2 \sin\left(\frac{1}{x}\right)$.





Note that for all x: $-1 \le \sin\left(\frac{1}{x}\right) \le 1$.

Multiply everywhere by $x^2 \ge 0$ and the inequalities do not change:

$$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$$

Now $\lim\limits_{x\to0} -x^2 = 0 = \lim\limits_{x\to0} x^2$

Hence by the Sandwich Theorem we conclude

$$\lim_{x\to0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

See the pictures on previous page!

Fundamental Trigonometric Limits

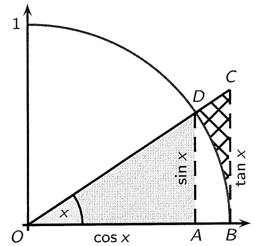
The following two trigonometric limits are important for developing the differential calculus for trigonometric functions:

$$\lim_{x\to0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to0} \frac{1-\cos x}{x} = 0$$

- Note that the angle x is measured in radians.
- We will prove both statements.
- The proof of the first statement uses a nice geometric argument and the sandwich theorem.
- The second statement follows from the first.

Proof that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

- Since we are interested in the limit as $x \to 0$, we can restrict the values of x to values close to 0.
- We split the proof into two cases, one in which $0 < x < \pi/2$, the other in which $-\pi/2 < x < 0$.
- Since $f(x) = \sin x/x$ is an even function (indeed, it is the quotient of two odd functions!) we only need to study the case $0 < x < \pi/2$.



In this case, both x and $\sin x$ are positive.

We draw the unit circle together with the triangles OAD and OBC. The angle x is measured in radians. Since $\overline{OB} = 1$, we find that

arc length of $BD = x$ $\quad \overline{OA} = \cos x \quad \overline{AD} = \sin x \quad \overline{BC} = \tan x$.

Furthermore the picture illustrates that

area of $OAD \le$ area of sector $OBD \le$ area of OBC

The area of a sector of central angle x (in radians) and radius r is $\frac{1}{2}r^2 x$.

Therefore, $\quad \dfrac{1}{2}\cos x \cdot \sin x \le \dfrac{1}{2} \cdot 1^2 \cdot x \le \dfrac{1}{2} \cdot 1 \cdot \tan x$.

Dividing this pair of inequalities by $1/2 \sin x$ yields

$$\cos x \le \frac{x}{\sin x} \le \frac{1}{\cos x}.$$

Solving now for $\sin x/x$ we obtain

$$\cos x \le \frac{\sin x}{x} \le \frac{1}{\cos x}.$$

We can now take the limit as $x \to 0^+$ and find that

$$\lim_{x\to 0^+} \cos x = 1 \qquad \lim_{x\to 0^+} \frac{1}{\cos x} = 1.$$

Finally the Sandwich Theorem yields $\quad \lim\limits_{x\to 0^+} \dfrac{\sin x}{x} = 1.$

By symmetry we also have that $\lim\limits_{x\to 0^-} \dfrac{\sin x}{x} = 1.$

Proof that $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x} = 0$

Multiplying both numerator and denominator of $f(x) = (1-\cos x)/x$ by $1+\cos x$, we can reduce the second statement to the first:

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos x}{x} &= \lim_{x\to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x} \\
&= \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} \\
&= \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)} \\
&= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\sin x}{1+\cos x} \\
&= 1 \cdot 0 = 0
\end{aligned}$$

Example 3: (Online Homework HW10, # 7)

Evaluate $\quad \lim\limits_{\theta\to 0} \dfrac{\sin(4\theta)\sin(8\theta)}{\theta^2}$.

Using a different letter we have

$$\lim_{u\to 0} \frac{\sin(u)}{u} = 1$$

Hence we can rewrite our limit as:

$$\lim_{\theta\to 0} \frac{\sin(4\theta)\sin(8\theta)}{\theta^2} = \lim_{\theta\to 0}\left[\frac{\sin(4\theta)}{\theta}\right]\left[\frac{\sin(8\theta)}{\theta}\right]$$

We need to have both terms of the form $\frac{\sin(u)}{u}$

Hence we adjust as follows:

$$\lim_{\theta\to 0}\left[\frac{\sin(4\theta)}{4\theta}\right]\left[\frac{\sin(8\theta)}{8\theta}\right]\cdot 32 =$$

$$= 32\left[\lim_{\underbrace{\theta\to 0}_{\text{also } 4\theta\to 0}} \frac{\sin 4\theta}{4\theta}\right]\cdot\left[\lim_{\underbrace{\theta\to 0}_{\text{also } 8\theta\to 0}} \frac{\sin(8\theta)}{8\theta}\right] = 32\cdot 1\cdot 1 = \boxed{32}$$

Example 4: (Online Homework HW10, # 10)

Evaluate $\lim_{x\to 0} \frac{\tan(5x)}{\tan(6x)}$.

$$\lim_{x\to 0}\frac{\tan(5x)}{\tan(6x)} = \lim_{x\to 0} \frac{\sin(5x)}{\cos(5x)}\cdot\frac{1}{\frac{\sin(6x)}{\cos(6x)}} =$$

$$= \lim_{x\to 0}\frac{\sin(5x)}{\sin(6x)}\cdot\frac{\cos(6x)}{\cos(5x)} =$$ direct substitution would give $\frac{0}{0}\cdot\frac{1}{1} = \frac{0}{0}$

But divide top and bottom by x:

$$= \lim_{x\to 0}\left\{\left[\frac{\sin(5x)}{x}\cdot\frac{x}{\sin(6x)}\right]\cdot\frac{\cos(6x)}{\cos(5x)}\right\}$$

= as before we re-adjust the coefficients:

$$= \lim_{x\to 0}\left\{\left[\frac{\sin(5x)}{5x}\cdot\frac{6x}{\sin(6x)}\right]\cdot\frac{5}{6}\cdot\frac{\cos(6x)}{\cos(5x)}\right\} =$$

$$= \frac{5}{6}\cdot\left[\underbrace{\lim_{x\to 0}\frac{\sin(5x)}{5x}}_{\to 1}\right]\left[\underbrace{\lim_{x\to 0}\frac{6x}{\sin(6x)}}_{\to 1}\right]\cdot\underbrace{\lim_{x\to 0}\frac{\cos(6x)}{\cos(5x)}}_{\to 1} = \boxed{\frac{5}{6}}$$

Example 5: (Neuhauser, Example 1(c), p. 126)

Evaluate $\lim_{x\to 0}\frac{\sec x - 1}{x\sec x}$.

$$\lim_{x\to 0}\frac{\sec x-1}{x\sec x}=\lim_{x\to 0}\frac{\frac{1}{\cos x}-1}{x\cdot\frac{1}{\cos x}}=$$

$$=\lim_{x\to 0}\frac{\frac{1-\cos x}{\cos x}}{\frac{x}{\cos x}}=\lim_{x\to 0}\frac{1-\cos x}{x}=0$$

by the fundamental limit we showed earlier.

Example 6: (Online Homework HW10, # 13)

Evaluate $\displaystyle\lim_{x\to\pi/4}\frac{3(\sin x-\cos x)}{5\cos(2x)}$.

$$\lim_{x\to \pi/4}\frac{3(\sin x-\cos x)}{5\cos(2x)}=\frac{3\left[\sin(\pi/4)-\cos(\pi/4)\right]}{5\cos\left(2\cdot\frac{\pi}{4}\right)}$$

substitution Theorem since trig functions are continuous

$$=\frac{3\left[\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\right]}{5\cos(\pi/2)}=\frac{0}{0}$$

What do we do? Try double angle formula

$$\lim_{x\to\pi/4}\frac{3(\sin x-\cos x)}{5\left[\cos^2 x-\sin^2 x\right]}=\lim_{x\to\frac{\pi}{4}}\frac{3(\sin x-\cos x)}{5\left[\cos x-\sin x\right]\cdot\left[\cos x+\sin x\right]}$$

can factor it!

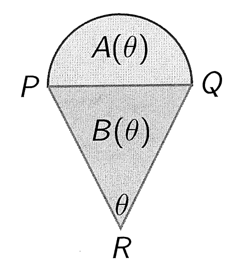
Simplify (careful to the sign!)

$$=\lim_{x\to\frac{\pi}{4}}\frac{-3}{5\left[\cos x+\sin x\right]}=\frac{-3}{5\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right)}=\boxed{-\frac{3}{5\sqrt{2}}\cong -0.424264}$$

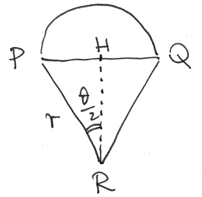
Example 7: (Online Homework HW10, # 14)

A semicircle with diameter PQ sits on an isosceles triangle PQR to form a region shaped like an ice cream cone, as shown in the figure. If $A(\theta)$ is the area of the semicircle and $B(\theta)$ is the area of the triangle, find

$$\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)}$$



We need the height and base of the triangle and the radius of the circle:



Use right-triangle trigonometry

$\overline{RH} = r\cos(\theta/2)$

$\overline{PH} = r\sin(\theta/2)$

hence $PQ = 2r\sin\left(\frac{\theta}{2}\right)$

Area of triangle:

$$\overset{\triangle}{PRQ} = B(\theta) = \frac{1}{2}\,\overline{PQ}\cdot\overline{RH} = \frac{1}{2}\left[2r\sin\left(\frac{\theta}{2}\right)\right]\left[r\cos\left(\theta/2\right)\right]$$

Area semicircle:

$$A(\theta) = \frac{1}{2}\pi\left(\overline{PH}\right)^2 = \frac{1}{2}\pi r^2\sin^2\left(\frac{\theta}{2}\right)$$

Hence

$$\frac{A(\theta)}{B(\theta)} = \frac{\frac{1}{2}\pi \cancel{r^2}\sin^{\cancel{2}}(\theta/2)}{\cancel{r^2}\cancel{\sin\left(\frac{\theta}{2}\right)}\cos\left(\frac{\theta}{2}\right)} = \frac{1}{2}\pi\cdot\frac{\sin(\theta/2)}{\cos(\theta/2)} = \frac{1}{2}\pi\tan(\theta/2)$$

Hence

$$\lim_{\theta\to 0}\frac{A(\theta)}{B(\theta)} = \lim_{\theta\to 0}\frac{1}{2}\pi\tan(\theta/2) = 0$$

as tan is a continuous function and $\underline{\tan(0)=0}$.

Trigonometric and Exponential Functions

- We will sometimes use the double angle formulas

$$\begin{aligned}\cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha \\ &= 2\cos^2\alpha - 1 \\ &= 1 - 2\sin^2\alpha\end{aligned} \quad \text{and} \quad \sin(2\alpha) = 2\sin\alpha\cos\alpha$$

which are special cases of the following addition formulas

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

- What about $\sin(\alpha/2)$ and $\cos(\alpha/2)$? With some work

$$\cos(\alpha/2) = \pm\sqrt{\frac{1+\cos\alpha}{2}} \qquad \sin(\alpha/2) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

(the sign ($+$ or $-$) depends on the quadrant in which $\frac{\alpha}{2}$ lies.)

- Is there a 'simple' way of remembering the above formulas?

Euler's Formula

Euler's formula states that, for any real number x,

$$\boxed{e^{ix} = \cos x + i\sin x,}$$

where i is the imaginary unit ($i^2 = -1$).

- For any α and β, using Euler's formula, we have

$$\begin{aligned}\cos(\alpha+\beta) + i\sin(\alpha+\beta) &= e^{i(\alpha+\beta)} \\ &= e^{i\alpha}\cdot e^{i\beta} \\ &= (\cos\alpha + i\sin\alpha)\cdot(\cos\beta + i\sin\beta) \\ &= (\cos\alpha\cos\beta + i^2\sin\alpha\sin\beta) \\ &\quad + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta).\end{aligned}$$

- Thus, by comparing the terms, we obtain

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

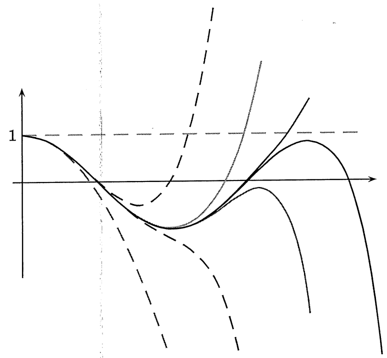
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Approximating $\cos x$

Consider the graph of the polynomial

$$T_{2n}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^{n-1}\frac{x^{2(n-1)}}{(2n-2)!} + (-1)^n\frac{x^{2n}}{(2n)!}.$$

As n increases, the graph of $T_{2n}(x)$ appears to approach the one of $\cos x$. This suggests that we can approximate $\cos x$ with $T_{2n}(x)$ as $n \to \infty$.



$y = 1$

$y = 1 - \frac{x^2}{2!}$

$y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

$y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$

$y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$

$y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}$

$y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!}$

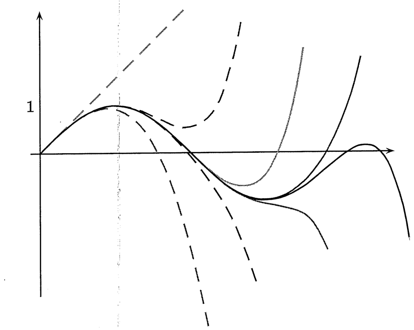
$y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \frac{x^{14}}{14!}$

Approximating $\sin x$

Consider the graph of the polynomial

$$T_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + (-1)^n\frac{x^{2n+1}}{(2n+1)!}.$$

As n increases, the graph of $T_{2n+1}(x)$ appears to approach the one of $\sin x$. This suggests that we can approximate $\sin x$ with $T_{2n+1}(x)$ as $n \to \infty$.



$y = x$

$y = x - \frac{x^3}{3!}$

$y = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$

$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$

$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!}$

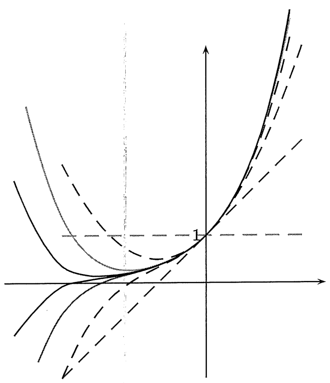
$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!}$

Approximating e^x

Consider the graph of the polynomial

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}.$$

As n increases, the graph of $T_n(x)$ appears to approach the one of e^x. This suggests that we can approximate e^x with $T_n(x)$ as $n \to \infty$.



$y = 1$

$y = 1 + x$

$y = 1 + x + \frac{x^2}{2!}$

$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$

$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}$

$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}$

Idea of Why Euler's Formula Works

To justify Euler's formula, we use the polynomial approximations for e^x, $\cos x$ and $\sin x$ that we just discussed. We start by approximating e^{ix}:

$$\begin{aligned}
e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \frac{(ix)^8}{8!} + \cdots \\
&= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \frac{x^8}{8!} + \cdots \\
&= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\
&= \cos x + i \sin x
\end{aligned}$$

Curiosity: From Euler's formula with $x = \pi$ we obtain

$$e^{i\pi} + 1 = 0$$

which involves five interesting math values in one short equation.