MA 137 – Calculus 1 with Life Science Applications
Formal Definition of the Derivative
(Sections 4.1 & 4.2)

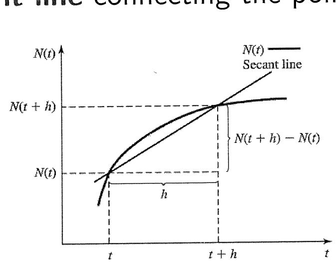
Department of Mathematics
University of Kentucky

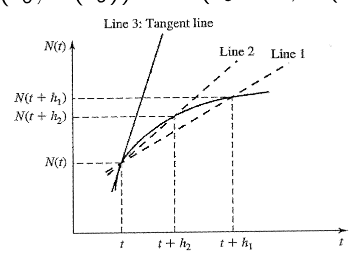
Average Growth Rate

- Population growth in populations with discrete breeding seasons (as discussed in Chapter 2) can be described by the change in population size from generation to generation.
- By contrast, in populations that breed continuously, there is no natural time scale such as generations. Instead, we will look at how the population size changes over small time intervals.
- We denote the population size at time t by $N(t)$, where t is now varying continuously over the interval $[0,\infty)$. We investigate how the population size changes during the interval $[t_0, t_0+h]$, where $h>0$. The *absolute change* during this interval, denoted by ΔN, is
$$\Delta N = N(t_0+h) - N(t_0).$$
- To obtain the *relative change* during this interval, we divide ΔN by the length of the interval, denoted by Δt, which is h. We find that
$$\frac{\Delta N}{\Delta t} = \frac{N(t_0+h)-N(t_0)}{h}.$$
This ratio is called the **average growth rate**.

Geometric Interpretation

We see from the picture below [left] that $\Delta N/\Delta t$ is the slope of the **secant line** connecting the points $(t_0, N(t_0))$ and $(t_0+h, N(t_0+h))$.





Observe that the average growth rate $\Delta N/\Delta t$ depends on the length of the interval Δt.

This dependency is illustrated in the picture above [right], where we see that the slopes of the two secant lines (lines 1 and 2) are different.

But we also see that, as we choose smaller and smaller intervals, the secant lines converge to the **tangent line** at the point $(t_0, N(t_0))$ of the graph of $N(t)$ (line 3).

Instantaneous Growth Rate

The slope of the tangent line is called the **instantaneous growth rate** (at t_0) and is a convenient way to describe the growth of a continuously breeding population.

To obtain this quantity, we need to take a limit; that is, we need to shrink the length of the interval $[t_0, t_0+h]$ to 0 by letting h tend to 0. We express this operation as
$$\lim_{\Delta t\to 0}\frac{\Delta N}{\Delta t} = \lim_{h\to 0}\frac{N(t_0+h)-N(t_0)}{h}.$$

In the expression above, we take a limit of a quantity in which a continuously varying variable, namely, h, approaches some fixed value, namely, 0.

We denote the limiting value of $\Delta N/\Delta t$ as $\Delta t \to 0$ by $N'(t_0)$ (read "N prime of t_0") and call this quantity **the derivative of** $N(t)$ **at** t_0....provided that this limit exists!

The Derivative of a Function

We formalize the previous discussion for any function f.

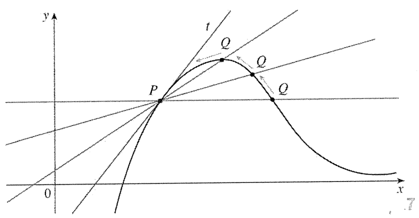
The **average rate of change** of the function $y = f(x)$ between $x = x_0$ and $x = x_1$ is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

By setting $h = x_1 - x_0$, i.e., $x_1 = x_0 + h$, the above expression becomes

$$\frac{\Delta f}{\Delta x} = \frac{f(x_0 + h) - f(x_0)}{h}$$

Those quantities represent the slope of the secant line that passes through the points $P(x_0, f(x_0))$ and $Q(x_1, f(x_1))$ [or $P(x_0, f(x_0))$ and $Q(x_0{+}h, f(x_0{+}h))$, respectively].



The instantaneous rate of change is defined as the result of computing the average rate of change over smaller and smaller intervals.

Definition

The **derivative of a function** f **at** x_0, denoted by $f'(x_0)$, is

$$f'(x_0) = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided that the limit exists.

In this case we say that the function f is **differentiable at** x_0.

Geometrically $f'(x_0)$ respresents the **slope of the tangent line**.

Note: To save on indices, we can also write $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ to denote the derivative of f at the point c.

- Now just drop the subscript 0 from the x_0 in the previous derivative formula, and you obtain the instantaneous rate of change of f with respect to x at a general point x. This is called the **derivative of** f **at** x and is denoted with $f'(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}.$$

  It is a function of x...no longer a number!
- We say that f is **differentiable** on an open interval (a, b) if $f'(x)$ exists at every $x \in (a, b)$.
- **Notations:** There is more than one way to write the derivative of a function $y = f(x)$. The following expressions are equivalent:

$$y' = \frac{dy}{dx} = f'(x) = \frac{df}{dx} = \frac{d}{dx} f(x).$$

  The notation $\frac{df}{dx}$ goes back to Leibniz and is called **Leibniz notation**.

  We can also write $\left.\frac{df}{dx}\right|_{x=x_0}$ to denote $f'(x_0)$.

Example 1: (Online Homework HW11, # 3)

Let $f(x)$ be the function $12x^2 - 2x + 11$. Then the difference quotient

$$\frac{f(1+h) - f(1)}{h}$$

can be simplified to $ah + b$ for $a =$ ______ and $b =$ ______.

Compute $\displaystyle \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}.$

* $f(x) = 12x^2 - 2x + 11$

* $\dfrac{f(1+h) - f(1)}{h} = \dfrac{\left[12(1+h)^2 - 2(1+h) + 11\right] - \left[12(1)^2 - 2(1) + 11\right]}{h}$

$$= \frac{12(1+2h+h^2) \cancel{-2} - 2h \cancel{+11} - 12 \cancel{+2} \cancel{-11}}{h}$$

$$= \frac{\cancel{12} + 24h + 12h^2 - 2h \cancel{-12}}{h} = \frac{22h + 12h^2}{h}$$

$$= \underline{\underline{22 + 12h}}$$

* $\displaystyle\lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \left[22 + 12h\right] = \underline{\underline{22}}$

Example 2:

If $f(x) = ax^2 + bx + c$, find $f'(x)$, using the definition of derivative. (a, b, and c are constants.)

* $f(x) = ax^2 + bx + c$

* $\dfrac{f(x+h) - f(x)}{h} = \dfrac{\left\{a[x+h]^2 + b[x+h] + c\right\} - \left\{ax^2 + bx + c\right\}}{h}$

$$= \frac{\cancel{ax^2} + 2axh + ah^2 \cancel{+bx} + bh \cancel{+c} \cancel{-ax^2} \cancel{-bx} \cancel{-c}}{h}$$

$$= \frac{2axh + bh + ah^2}{h} = \frac{\cancel{h}(2ax + b + ah)}{\cancel{h}}$$

$$= \boxed{2ax + b + ah}$$

* $\displaystyle\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \left(2ax + b + \overset{\to 0}{ah}\right) = \boxed{2ax + b}$

Thus: $f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$ goes to zero

In particular, if $a = 0$; $f(x) = bx + c$ is such that $f'(x) = b$. Hence the derivative of a linear function is its slope.

Equation of the Tangent Line at a Point

If the derivative of a function f exists at $x = x_0$, then $f'(x_0)$ is the slope of the tangent line at the point $P(x_0, f(x_0))$.
The equation of the tangent line to the graph of f at P is given by

$$y - f(x_0) = f'(x_0)(x - x_0).$$

The importance of computing the equation of the tangent line to the graph of a function f at a point $P(x_0, f(x_0))$ lies in the fact that if we look at a portion of the graph of f near the point P, it becomes indistinguishable from the tangent line at P.

In other words, the values of the function are close to those of the linear function whose graph is the tangent line.

For this reason, the linear function whose graph is the tangent line to $y = f(x)$ at the point $P(x_0, f(x_0))$ is called the **linear approximation** of f near $x = x_0$.

Example 3: (Online Homework HW11, # 8)

If $f(x) = 4x + \frac{4}{x}$, find $f'(2)$, using the definition of derivative.

Use this to find the equation of the tangent line to the graph of $y = f(x)$ at the point $(2, f(2))$.

$f(x) = 4x + \frac{4}{x}$

In order to find the tangent line at $x=2$ we need $P(2, f(2)) = (2, 10)$ as $f(2) = 4\cdot 2 + \frac{4}{2} = 8+2 = 10$

and the slope of the tg. line $\boxed{f'(2)}$

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{\left[4(2+h) + \frac{4}{2+h}\right] - 10}{h}$$

$$= \lim_{h\to 0} \frac{4(2+h)^2 + 4 - 10(2+h)}{h(2+h)}$$

$$= \lim_{h\to 0} \frac{\cancel{16} + 16h + 4h^2 + \cancel{4} - \cancel{20} - 10h}{h(2+h)}$$

$$= \lim_{h\to 0} \frac{6h + 4h^2}{h(2+h)} = \lim_{h\to 0} \frac{\cancel{h}(6+4h)}{\cancel{h}(2+h)}$$

$$= \lim_{h\to 0} \frac{6+4h}{2+h} = \frac{6}{2} = \boxed{3}$$

Hence $f'(2) = 3$. Therefore the equation of the tangent line at $P(2, 10)$ is

$$\boxed{y - 10 = 3(x-2)}$$

or

$$\boxed{y = 3x + 4}$$

Example 4:

If $f(x) = \sqrt{x}$, find $f'(x)$, using the definition of derivative.

* $\boxed{f(x) = \sqrt{x}}$

* $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$

$$= \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h\to 0} \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{\cancel{x} + h \cancel{-x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} =$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}} \longleftarrow \underline{\text{derivative}}$$

Example 5: (Online Homework HW11, # 11)

Assume that $f(x)$ is everywhere continuous and it is given to you that

$$\lim_{x\to 7} \frac{f(x) + 9}{x - 7} = 10.$$

It follows that $y =$ ____________ is the equation of the tangent line to $y = f(x)$ at the point (____,____).

Recall that we can also write $f'(c)$ as:

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$

Thus if we consider

$$10 = \lim_{x\to 7} \frac{f(x) + 9}{x - 7} = \lim_{x\to 7} \frac{f(x) - (-9)}{x - 7}$$

we have that $\boxed{c = 7}$ $\quad f(c) = f(7) = -9$

and $f'(c) = f'(7) = 10$. Thus the equation of the tangent line at $P(7, -9)$ is

$$\boxed{y - (-9) = 10(x - 7)} \quad \text{or} \quad \boxed{y = 10x - 79}$$

Example 6: (Online Homework HW11, # 4)

The limit below represents a derivative $f'(a)$.

$$\lim_{h\to 0} \frac{(-4+h)^3+64}{h}.$$

Find $f(x)$ and a.

$$f'(a) = \lim_{h\to 0} \frac{(-4+h)^3+64}{h}$$

$$= \lim_{h\to 0} \frac{(-4+h)^3-(-64)}{h}$$

$$= \lim_{h\to 0} \frac{(-4+h)^3-(-4)^3}{h}$$

$$\therefore \quad \boxed{f(x) = x^3} \qquad \boxed{a=-4}$$

Differentiability and Continuity

A function f is differentiable at a point if the derivative at that point exists. That is, if the tangent line at that point is well defined.

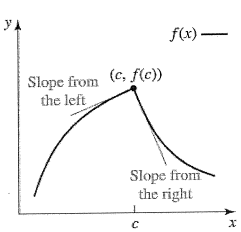
There are two ways that a tangent line might not exist.
It depends on how limits fail to exist:

(a) left-hand and right-hand limit do not agree;
(b) one of these limits is infinite.

Continuity alone is not enough for a function to be differentiable:

(a) The function $f(x) = |x|$ is continuous at all values of x, but it is not differentiable at $x=0$. It has a **sharp corner** at $x=0$

(b) The function $f(x) = x^{1/3}$ is continuous for all x, but it is not differentiable at $x=0$. There is a **vertical tangent line** at $x=0$.



Differentiability Implies Continuity

However, if a function is differentiable, it is also continuous.

Theorem

If f is differentiable at $x=x_0$, then f is also continuous at $x=x_0$.

Proof: To show that f is continuous at $x=x_0$, we must show that

$$\lim_{x\to x_0} f(x) = f(x_0) \qquad \text{or} \qquad \lim_{x\to x_0}[f(x)-f(x_0)] = 0.$$

However

$$\begin{aligned}\lim_{x\to x_0}[f(x)-f(x_0)] &= \lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0}\cdot(x-x_0)\right] \\ &= \lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0}\right]\cdot \lim_{x\to x_0}(x-x_0) \\ &= f'(x_0)\cdot \lim_{x\to x_0}(x-x_0) \\ &= f'(x_0)\cdot 0 \\ &= 0.\end{aligned}$$

Example 7: (Online Homework HW11, # 14)

Find a and b so that the function

$$f(x) = \begin{cases} x^2 - 2x + 3 & \text{if } x \le 2 \\ ax^2 + 6x + b & \text{if } x > 2 \end{cases}$$

is both continuous and differentiable.

$f(x)$ is made of two pieces of parabolas. These are continuous and differentiable for $x<2$ and $x>2$. The problem is at $x=2$.

We need to make sure that f is continuous at $x=2$ and that the derivative exists at $x=2$.

(1.) for the continuity we need: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$

Thus: $\lim_{x\to 2^-} (x^2-2x+3) \underset{\text{MUST}}{=} \lim_{x\to 2^+} (ax^2+6x+b)$

i.e. $2^2 - 2(2) + 3 = a(2)^2 + 6(2) + b$

$\iff \boxed{4a + b = -9}$

(2) The derivative of f is

$$f'(x) = \begin{cases} 2x - 2 & \text{if } x < 2 \\ 2ax + 6 & \text{if } x > 2 \end{cases}$$

We need to make sure that it exists for $x=2$

$\lim_{x\to 2^-} f'(x) = \lim_{x\to 2^+} f'(x)$; hence

$2(2) - 2 \underset{\text{MUST}}{=} 2a(2) + 6$

$\iff 2 = 4a + 6 \iff 4a = -4 \iff \boxed{a = -1}$

Hence: $\begin{cases} a = -1 \\ 4a + b = -9 \end{cases} \implies \boxed{b = -5}$