MA 137 – Calculus 1 with Life Science Applications
The Chain Rule and Higher Derivatives
(Sections 4.5 & 4.7)

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The Chain Rule

Theorem

If g is differentiable at x and f is differentiable at $y = g(x)$, then the composite function $(f \circ g)(x) = f[g(x)]$ is differentiable at x, and the derivative is given by

$$(f \circ g)'(x) = f'[g(x)] \cdot g'(x)$$

- The proof of the theorem is on p. 164 of the Neuhauser's textbook.
- The function g is the inner function; the function f is the outer function.
- The expression $f'[g(x)] \cdot g'(x)$ thus means that we need to find the derivative of the outer function, evaluated at $g(x)$, and the derivative of the inner function, evaluated at x, and then multiply the two together.
- A special case of the chain rule is called the **power chain rule**:

If $\quad y = [f(x)]^n \quad$ then $\quad \dfrac{dy}{dx} = n[f(x)]^{n-1} \cdot f'(x)$

The Chain Rule in Leibniz Notation

The derivative of $f \circ g$ can be written in Leibniz notation.

If we set $u = g(x)$, then

$$\begin{aligned} \frac{d}{dx}[(f \circ g)(x)] &= \frac{d}{dx} f[g(x)] \\ &\overset{u=g(x)}{=} \frac{d}{dx} f(u) \\ &= \frac{df}{du} \cdot \frac{du}{dx} \end{aligned}$$

This form of the chain rule emphasizes that, in order to differentiate $f \circ g$, we multiply the derivative of the outer function and the derivative of the inner function, the former evaluated at u, the latter at x.

Example 1: (Online Homework HW13, # 3)

Let $F(x) = f(f(x))$ and $G(x) = (F(x))^2$ and suppose that

$$f(5) = 3 \qquad f(7) = 5 \qquad f'(5) = 8 \qquad f'(7) = 13$$

Find $F'(7)$ and $G'(7)$.

$f(5)=3 \qquad f(7)=5 \qquad f'(5)=8 \qquad f'(7)=13$

(1) $\boxed{F(x)=f(f(x))}$

By the chain rule $F'(x)=f'(f(x))\cdot f'(x)$

Thus $F'(7)=f'(f(7))\cdot f'(7)=f'(5)\cdot f'(7)$

$= 8\cdot 13 = \underline{\underline{104}}$

(2) $\boxed{G(x)=(F(x))^2}$

By the (power) chain rule $G'(x)=2F(x)\cdot F'(x)$

Thus $G'(7)=2F(7)\cdot F'(7)=2\cdot 3\cdot 104=\underline{\underline{624}}$

$F(7)=f(f(7))=f(5)=3$

Example 2: (Online Homework HW13, # 6)

Let $f(x)=\dfrac{9}{(2x^2-3x+6)^4}$. Find $f'(x)$.

$$f(x)=\frac{9}{(2x^2-3x+6)^4}=9(2x^2-3x+6)^{-4}$$

Hence, by the (power) chain rule

$$f'(x)=9\cdot(-4)(2x^2-3x+6)^{-4-1}\cdot(4x-3)$$

$$=\boxed{\frac{-36(4x-3)}{(2x^2-3x+6)^5}}$$

The Quotient Rule Using the Chain Rule

We can prove quotient rule using the product and (power) chain rules. Treat the quotient f/g as a product of f and the reciprocal of g. I.e.,

$$\frac{f(x)}{g(x)}=f(x)\cdot g(x)^{-1}.$$

Next, apply the product rule

$$\left(\frac{f(x)}{g(x)}\right)'=[f(x)\cdot g(x)^{-1}]'=f'(x)\cdot g(x)^{-1}+f(x)\cdot[g(x)^{-1}]'$$

and apply the (power) chain rule to find $[g(x)^{-1}]'$. We obtain

$$=f'(x)\cdot[g(x)^{-1}]+f(x)\cdot[(-1)g(x)^{-2}\cdot g'(x)].$$

Finish by writing the expression with a common denominator of $[g(x)]^2$

$$=\frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{[g(x)]^2}.$$

Example 3: (Neuhauser, Example # 5, p. 170)

Find the derivative of $h(x) = \left(\dfrac{x}{x+1}\right)^2$.

$$h(x) = \left(\frac{x}{x+1}\right)^2$$

By the (power) chain rule

$$h'(x) = 2\left(\frac{x}{x+1}\right)^{2-1} \cdot \frac{d}{dx}\left(\frac{x}{x+1}\right)$$

$$= 2 \cdot \frac{x}{x+1} \cdot \frac{1\cdot(x+1) - x(1)}{(x+1)^2}$$

$$= \boxed{2 \cdot \frac{x}{(x+1)^3}}$$

Example 4: (Neuhauser, Problem # 32, p. 173)

Differentiate $g(N) = \dfrac{N}{(k+bN)^3}$ with respect to N.

Assume that b and k are positive constants.

$$g(N) = \frac{N}{(k+bN)^3}$$

Use the quotient rule and the chain rule:

$$g'(N) = \frac{1\cdot(k+bN)^3 - N\cdot 3(k+bN)^{3-1}\cdot(b)}{\left[(k+bN)^3\right]^2}$$

$$= \frac{(k+bN)^3 - 3bN(k+bN)^2}{(k+bN)^6}$$

$$= \frac{(k+bN)^2\left[k+bN-3bN\right]}{(k+bN)^6}$$

$$= \boxed{\frac{k-2bN}{(k+bN)^4}}$$

Example 5: (Neuhauser, Problem # 39, p. 174)

Find the derivative of

$$\frac{[f(x)]^2}{g(2x)+2x}$$

assuming that f and g are both differentiable functions.

$$y = \frac{[f(x)]^2}{g(2x)+2x}$$

We find y' using the quotient rule and the chain rule:

$$y' = \frac{\{[f(x)]^2\}' \cdot (g(2x)+2x) - [f(x)]^2 \cdot \{g(2x)+2x\}'}{(g(2x)+2x)^2}$$

$$= \frac{2f(x)\cdot f'(x)\cdot(g(2x)+2x) - [f(x)]^2\cdot(g'(2x)\cdot 2+2)}{(g(2x)+2x)^2}$$

$$= \boxed{\frac{2f(x)\cdot[f'(x)(g(2x)+2x) - f(x)(g'(2x)+1)]}{[g(2x)+2x]^2}}$$

Higher Derivatives

- The derivative of a function f is itself a function. We refer to this derivative as the **first derivative**, denoted f'. If the first derivative exists, we say that the function is once differentiable.
- Given that the first derivative is a function, we can define its derivative (where it exists). This derivative is called the **second derivative** and is denoted f''. If the second derivative exists, we say that the original function is twice differentiable.
- This second derivative is again a function; hence, we can define its derivative (where it exists). The result is the **third derivative**, denoted f'''. If the third derivative exists, we say that the original function is three times differentiable.
- We can continue in this manner; from the fourth derivative on, we denote the derivatives by $f^{(4)}, f^{(5)}$, and so on.
  If the nth derivative exists, we say that the original function is n times differentiable.

- Polynomials are functions that can be differentiated as many times as desired. The reason is that the first derivative of a polynomial of degree n is a polynomial of degree $n-1$. Since the derivative is a polynomial as well, we can find its derivative, and so on. Eventually, the derivative will be equal to 0.
- We can write higher-order derivatives in Leibniz notation: The nth derivative of $f(x)$ is denoted by

$$\frac{d^n f}{dx^n}$$

Example 6: (Online Homework HW13, # 4)

Find the first and second derivatives of the following function

$$f(x) = (5 - 3x^2)^4$$

$$f(x) = (5 - 3x^2)^4$$

Then:

$$f'(x) = 4(5-3x^2)^3 \cdot (-6x)$$

$$= -24x(5-3x^2)^3$$

$$f''(x) = -24(5-3x^2)^3 - 24x \cdot \left[3(5-3x^2)^2 \cdot (-6x)\right]$$

$$= -24(5-3x^2)^3 + 18 \cdot 24x^2(5-3x^2)^2$$

$$= 24(5-3x^2)^2 \cdot \left[-(5-3x^2) + 18x^2\right]$$

$$= \boxed{24(5-3x^2)^2 \cdot (21x^2-5)}$$

Example 7: (Online Homework HW13, # 16)

Find the first and second derivatives of the following function

$$y = \frac{1-4u}{1+3u}$$

$$y = \frac{1-4u}{1+3u}$$

$$y' = \frac{-4(1+3u) - (1-4u)(3)}{(1+3u)^2}$$

$$= \frac{-4 \cancel{-12u} - 3 \cancel{+12u}}{(1+3u)^2} = \boxed{\frac{-7}{(1+3u)^2}}$$

$$= -7(1+3u)^{-2}$$

$$y'' = -7(-2)(1+3u)^{-2-1} \cdot (3)$$

$$= \boxed{\frac{42}{(1+3u)^3}}$$

Velocity and Acceleration

The velocity of an object that moves on a straight line is the derivative of the object's position. The derivative of the velocity is the acceleration.

If $s(t)$ denotes the position of an object moving on a straight line, $v(t)$ its velocity, and $a(t)$ its acceleration, then the three quantities are related as follows:

$$v(t) = \frac{ds}{dt} \quad \text{and} \quad a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}.$$

Example 8: (Neuhauser, Problem # 15, p. 184)

Neglecting air resistance, the height h (in meters) of an object thrown vertically from the ground with initial velocity v_0 is given by

$$h(t) = v_0 t - \frac{1}{2} g t^2$$

where $g = 9.81\text{m/s}^2$ is the earth's gravitational constant and t is the time (in seconds) elapsed since the object was released.

(a) Find the velocity and the acceleration of the object.

(b) Find the time when the velocity is equal to 0. In which direction is the object traveling right before this time? in which direction right after this time?

$$h(t) = v_0 t - \frac{1}{2} g t^2$$

(a) $v(t) = h'(t) = \frac{dh}{dt} = v_0 - \frac{1}{2} g \cdot 2t$

$= \boxed{v_0 - gt}$

$a(t) = v'(t) = h''(t) = \frac{d^2 h}{dt^2} = \boxed{-g}$

(b) $v(t) = 0 \iff v_0 - gt = 0 \iff \boxed{t = \frac{v_0}{g}}$

Before $\frac{v_0}{g}$ we have that $v(t)$ is positive so the object goes up; after $\frac{v_0}{g}$ the velocity is negative so the object goes down.