

MA 137 – Calculus 1 with Life Science Applications
Implicit Differentiation
 (Section 4.6)

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When we say that f is implicitly defined by the equation given in (1), we mean that the equation

$$x^3 + [f(x)]^3 = 6x f(x)$$

is true for all values of x in the domain of f .

Fortunately, there is a very useful technique, based on the chain rule, that will allow us to find dy/dx for implicitly defined functions.

This technique is called **implicit differentiation**.

We summarize the steps we take to find dy/dx when an equation defines y implicitly as a differentiable function of x :

1. Differentiate both sides of the equation with respect to x , keeping in mind that y is a function of x .

[Note: differentiating terms involving y typically requires the chain rule.]

2. Solve the resulting equation for dy/dx .

Implicit Differentiation

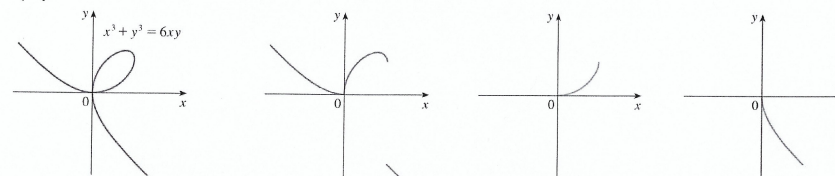
So far, we have considered only functions of the form $y = f(x)$, which define y explicitly as a function of x .

It is also possible to define y implicitly as a function of x , as in the following equation:

$$x^3 + y^3 = 6xy \quad (1)$$

Here, y is still given as a function of x (i.e., y is the dependent variable), but there is no obvious way to solve for y .

Below are the graphs of three such functions related to equation (1), dubbed the folium of Descartes.



Example 1:

- (a) Find y' if y is implicitly defined by $x^3 + y^3 = 6xy$.
- (b) Find an equation for the tangent line to the folium of Descartes $x^3 + y^3 = 6xy$ at the point $(3, 3)$.

(a) Consider $x^3 + y^3 = 6xy$. Take $\frac{d}{dx}$ of both sides:

$$\frac{d}{dx} [x^3 + y^3] = \frac{d}{dx} (6xy)$$

$$\Leftrightarrow \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 6 \frac{d}{dx}(xy)$$

$$\Leftrightarrow 3x^2 + \underbrace{\frac{d}{dy}(y^3) \cdot \frac{dy}{dx}}_{\text{chain rule}} = 6 \cdot y + \underbrace{6x \frac{dy}{dx}}_{\text{product rule}}$$

$$\Leftrightarrow 3x^2 + 3y^2 \cdot \frac{dy}{dx} = 6y + 6x \frac{dy}{dx}$$

$$\Leftrightarrow 3y^2 \frac{dy}{dx} - 6x \frac{dy}{dx} = 6y - 3x^2$$

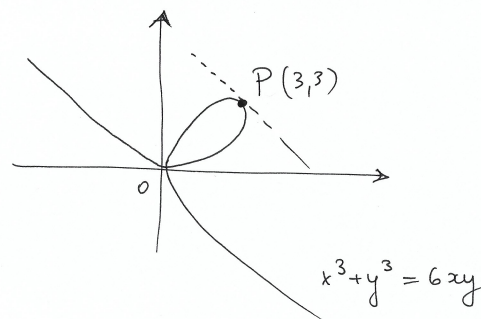
$$\Leftrightarrow \boxed{\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}}$$

(b) $\left. \frac{dy}{dx} \right|_{P(3,3)} = \frac{2(3) - 3^2}{3^2 - 2 \cdot 3} = \frac{6 - 9}{9 - 6} = \frac{-3}{3} = \boxed{-1}$

Hence the equation of the tangent line at $P(3,3)$ is

$$y - 3 = (-1)(x - 3)$$

$$\boxed{y = -x + 6}$$



Example 2: (Online Homework HW14, # 2)

Given $xy + 2x + 3x^2 = -4$:

(a) Find y' by implicit differentiation.

(b) Solve the equation for y and differentiate to get y' in terms of x .
(The answers should be consistent!)

(a) Given $xy + 2x + 3x^2 = -4$. Take $\frac{d}{dx}$ of both sides

$$\frac{d}{dx} [xy + 2x + 3x^2] = \frac{d}{dx} (-4)$$

$$\frac{d}{dx}(xy) + 2 \frac{d}{dx}(x) + 3 \frac{d}{dx}(x^2) = 0$$

$$1 \cdot y + x \frac{dy}{dx} + 2 + 3(2x) = 0$$

$$\boxed{\frac{dy}{dx} = \frac{-2 - 6x - y}{x}}$$

(b) $xy = -2x - 3x^2 - 4 \Rightarrow y = \frac{-2x - 3x^2 - 4}{x}$

So explicitly $\boxed{y = -2 - 3x - \frac{4}{x}}$

$$y' = -3 - 4(-1)x^{-2} = \boxed{-3 + \frac{4}{x^2}}$$

Example 3: (Neuhauser, Problem # 8, p. 179)Find dy/dx by implicit differentiation if

$$\frac{x}{xy+1} = 2xy.$$

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$$\frac{x}{(xy+1)} = 2xy \iff x = 2xy(xy+1)$$

$$\iff x = 2x^2y^2 + 2xy.$$

Take the derivative of both sides wrt. x :

$$\frac{d}{dx} [x] = \frac{d}{dx} [2x^2y^2 + 2xy]$$

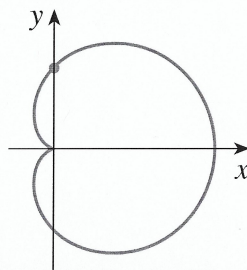
$$1 = 4xy^2 + 2x^2(2y \frac{dy}{dx}) + 2 \cdot 1 \cdot y + 2x \frac{dy}{dx}$$

$$1 - 4xy^2 - 2y = 4x^2y \frac{dy}{dx} + 2x \frac{dy}{dx}$$

$$\text{hence } \boxed{\frac{dy}{dx} = \frac{1 - 4xy^2 - 2y}{4x^2y + 2x}}$$

Example 4: (Online Homework HW14, # 6)Use implicit differentiation to find an equation of the tangent line to the curve (called **cardioid**)

$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2$$

at the point $(0, 1/2)$.

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$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2 \quad \text{cardioid}$$

We need the tangent line at $P(0, 1/2)$.We need the slope: so $\frac{dy}{dx} \Big|_P = ?$

We use implicit differentiation:

$$\frac{d}{dx} [x^2 + y^2] = \frac{d}{dx} [(2x^2 + 2y^2 - x)^2]$$

$$2x + \frac{d}{dx} (y^2) = 2(2x^2 + 2y^2 - x)^{2-1} \cdot \frac{d}{dx} (2x^2 + 2y^2 - x)$$

chain rule

$$2x + \underbrace{2y \cdot \frac{dy}{dx}}_{\text{chain rule}} = 2(2x^2 + 2y^2 - x) \cdot \left[4x + 4y \frac{dy}{dx} - 1 \right]$$

factor

$$2y \frac{dy}{dx} - 8y(2x^2 + 2y^2 - x) \frac{dy}{dx} = 2(2x^2 + 2y^2 - x)(4x - 1) - 2x$$

$$\frac{dy}{dx} (2y - 8y(2x^2 + 2y^2 - x)) = 2(2x^2 + 2y^2 - x)(4x - 1) - 2x$$

Simplify 2 on both sides

$$\therefore \frac{dy}{dx} = \frac{(2x^2 + 2y^2 - x)(4x - 1) - x}{y - 4y(2x^2 + 2y^2 - x)}$$

Evaluate the derivative when $x=0$ and $y=1/2$

$$\left. \frac{dy}{dx} \right|_{\mathbb{P}} = \frac{(2(\frac{1}{2})^2)(-1) - 0}{\frac{1}{2} - 4(\frac{1}{2})(2(\frac{1}{2})^2)} = \frac{-\frac{1}{2}}{\frac{1}{2} - 1}$$

$$= \frac{-\frac{1}{2}}{-\frac{1}{2}} = \underline{\underline{1}}$$

Thus the equation of the tg. line is

$$y - \frac{1}{2} = 1(x - 0) \quad \text{or} \quad y = x + \frac{1}{2}$$

Power Rule for Rational Exponents

We now provide a proof of the generalized form of the power rule when the exponent r is a rational number: $\frac{d}{dx}(x^r) = r x^{r-1}$.

We write $r = p/q$, where p and q are integers and are in lowest terms. (If q is even, we require x and y to be positive.) Then

$$y = x^r \iff y = x^{p/q} \iff y^q = x^p.$$

Differentiating both sides of $y^q = x^p$ with respect to x , we find that

$$qy^{q-1} \frac{dy}{dx} = px^{p-1}.$$

Hence

$$\begin{aligned} \frac{dy}{dx} &= \frac{p x^{p-1}}{q y^{q-1}} = \frac{p}{q} \frac{x^{p-1}}{(x^{p/q})^{q-1}} = \frac{p}{q} \frac{x^{p-1}}{x^{p-p/q}} = \frac{p}{q} x^{p-1-(p-p/q)} \\ &= \frac{p}{q} x^{p/q-1} = r x^{r-1} \end{aligned}$$