

MA 137 — Calculus 1 with Life Science Applications  
**Derivatives of Logarithmic Functions and  
 Logarithmic Differentiation**  
 (Section 4.10)

Department of Mathematics  
 University of Kentucky

1/12

## The Derivative of the Natural Logarithmic Function

### Theorem

The function  $\ln x$  is differentiable for all  $x > 0$ , and  $\frac{d}{dx} \ln x = \frac{1}{x}$ .

In particular, if  $g(x)$  is a differentiable function, it follows from the chain rule that

$$\frac{d}{dx} \ln g(x) = \frac{1}{g(x)} g'(x).$$

We can use the derivative of  $e^x$  and the relationship between the exponential and the natural logarithmic functions to find the derivative of the function  $\ln x$ . Namely, we start by taking the derivative with respect to  $x$  of both sides of  $e^{\ln x} = x$ . We obtain

$$\frac{d}{dx} e^{\ln x} = \frac{d}{dx} x \iff e^{\ln x} \frac{d}{dx} \ln x = 1 \iff \frac{d}{dx} \ln x = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

2/12

## Alternative Proof

We use the formal definition of the derivative and  $e^x = \lim_{u \rightarrow \infty} \left(1 + \frac{x}{u}\right)^u$

$$\begin{aligned} \frac{d}{dx} \ln x &\stackrel{\text{def}}{=} \lim_{h \rightarrow 0} \frac{\ln(x+h) - \ln x}{h} \\ &\stackrel{\text{ln prop.}}{=} \lim_{h \rightarrow 0} \frac{1}{h} \ln \left( \frac{x+h}{x} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{x} \frac{x}{h} \ln \left( 1 + \frac{1}{x/h} \right) \quad u = x/h \\ &\stackrel{\text{laws}}{=} \frac{1}{x} \lim_{u \rightarrow \infty} \ln \left( 1 + \frac{1}{u} \right)^u \\ &\stackrel{\text{cont.}}{=} \frac{1}{x} \ln \left[ \lim_{u \rightarrow \infty} \left( 1 + \frac{1}{u} \right)^u \right] \\ &= \frac{1}{x} \ln e = \frac{1}{x} \end{aligned}$$

3/12

## The Derivative of ANY Logarithmic Function

### Theorem

The function  $\log_a x$  is differentiable for  $x > 0$ , and  $\frac{d}{dx} \log_a x = \frac{1}{(\ln a)x}$ .

In particular, if  $g(x)$  is a differentiable function, it follows from the chain rule that

$$\frac{d}{dx} \log_a g(x) = \frac{1}{(\ln a)g(x)} g'(x).$$

From the base change formula for logarithms we have that

$$\log_a x = \frac{\ln x}{\ln a}$$

Thus it is enough to find the derivative of  $\ln x$ . Hence the formula.

4/12



**Example 1:** (Nuehauser, Problems # 28/34/52, p. 204)

Find  $\frac{dy}{dx}$  when  $y = \ln(1 - x^2)$ .

Find  $\frac{dy}{dx}$  when  $y = [\ln(1 - x^2)]^3$ .

Find  $\frac{dy}{ds}$  when  $y = \ln(\ln s)$ .

5/12

(a)  $y = \ln(1 - x^2)$

$$\frac{dy}{dx} = \frac{1}{1-x^2} \cdot (-2x) = \boxed{\frac{-2x}{1-x^2}}$$

chain rule

(b)  $y = [\ln(1 - x^2)]^3$

$$\begin{aligned} \frac{dy}{dx} &= 3 [\ln(1 - x^2)]^{3-1} \cdot (\ln(1 - x^2))' \\ &= 3 [\ln(1 - x^2)]^2 \cdot \frac{1}{1-x^2} \cdot (-2x) = \boxed{\frac{-6x [\ln(1 - x^2)]^2}{1-x^2}} \end{aligned}$$

(c)  $y = \ln(\ln s)$

$$\frac{dy}{ds} = \frac{1}{\ln s} \cdot (\ln s)' = \boxed{\frac{1}{\ln s} \cdot \frac{1}{s}}$$

**Example 2:** (Nuehauser, Problem # 56, p. 204)

Find  $\frac{dy}{dx}$  when  $y = \log(3x^2 - x + 2)$ .

[Note:  $\log = \log_{10}$ ]

6/12

$$y = \log(3x^2 - x + 2)$$

$$\frac{dy}{dx} = \frac{1}{\ln(10) \cdot (3x^2 - x + 2)} \cdot (6x - 1)$$

Recall the formula:

$$\left[ \frac{d}{dx} [\log_a g(x)] = \frac{1}{[\ln(a)] \cdot g(x)} \cdot g'(x) \right]$$

**Example 3:** (Nuehauser, Problem # 62, p. 204)

Assume that  $f(x)$  is differentiable with respect to  $x$ . Show that

$$\frac{d}{dx} \ln \left[ \frac{f(x)}{x} \right] = \frac{f'(x)}{f(x)} - \frac{1}{x}$$

$$y = \ln \left[ \frac{f(x)}{x} \right]$$

We want to compute  $\frac{dy}{dx}$ .

1<sup>st</sup> method

direct computation using the chain rule and the quotient rule:

$$\begin{aligned} y' &= \frac{1}{\frac{f(x)}{x}} \cdot \left[ \frac{f(x)}{x} \right]' = \frac{x}{f(x)} \cdot \frac{f'(x) \cdot x - f(x)}{x^2} \\ &= \frac{f'(x)x - f(x)}{x f(x)} = \frac{f'(x)}{f(x)} - \frac{1}{x} \end{aligned}$$

2<sup>nd</sup> method

$$y = \ln \left[ \frac{f(x)}{x} \right] = \ln[f(x)] - \ln x$$

Hence  $y' = \frac{1}{f(x)} \cdot f'(x) - \frac{1}{x} = \boxed{\frac{f'(x)}{f(x)} - \frac{1}{x}}$

**Logarithmic Differentiation**

In 1695, Leibniz introduced logarithmic differentiation, following Johann Bernoulli's suggestion to find derivatives of functions of the form

$$y = [f(x)]^x.$$

Bernoulli generalized this method and published his results two years later.

The **basic idea** is to take logarithms on both sides and then to use implicit differentiation.

**Example 4:** (Neuhauser, Example # 12, p. 202)

Find  $\frac{dy}{dx}$  when  $y = x^x$ .

What about  $\frac{d}{dx} [(2x)^{2x}]$ ?



$$y = x^x$$

to find  $y'$  we can take  $\ln$  of both sides:

$$\ln y = \ln x^x \quad \underline{\text{OR}} \quad \ln y = x \cdot \ln x$$

Hence, now take  $\frac{d}{dx}$  of both sides and use

the chain rule:

$$\frac{d}{dx} [\ln y] = \frac{d}{dx} [x \cdot \ln x]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = y [\ln x + 1] = \underline{\underline{x^x [\ln x + 1]}}$$

Alternative method:

$$y = x^x = e^{\ln x^x} = e^{\underline{x \ln x}}$$

$$\text{Hence } y' = e^{x \ln x} \cdot (x \ln x)'$$

$$= e^{x \ln x} \cdot (1 \cdot \ln x + x \cdot \frac{1}{x})$$

$$= e^{\ln x^x} \cdot (\ln x + 1)$$

$$= x^x \cdot (\ln x + 1) \quad \text{same answer as before !!!}$$

About  $y = (2x)^{2x}$

1<sup>st</sup> method:  $\ln y = \ln[(2x)^{2x}] \iff \ln y = 2x \cdot \ln(2x)$

Take the derivative:  $\frac{d}{dx} [\ln y] = \frac{d}{dx} [2x \cdot \ln(2x)]$

$$\iff \frac{1}{y} \cdot \frac{dy}{dx} = 2 \cdot \ln(2x) + 2x \cdot \left[ \frac{1}{2x} \cdot 2 \right]$$

$$\frac{dy}{dx} = y [2 \ln(2x) + 2] = \underline{\underline{(2x)^{2x} [2 \ln(2x) + 2]}}$$

2<sup>nd</sup> method:  $y = (2x)^{2x} = e^{\ln[(2x)^{2x}]} = e^{2x \cdot \ln(2x)}$

Hence  $y' = e^{2x \cdot \ln(2x)} \cdot [2x \cdot \ln(2x)]'$

$$= e^{\ln(2x)^{2x}} \cdot \left[ 2 \ln(2x) + 2x \cdot \left( \frac{1}{2x} \cdot 2 \right) \right]$$

$$= \underline{\underline{(2x)^{2x} \cdot [2 \ln(2x) + 2]}}$$

**Example 5:** (Neuhauser, Problems # 68/75/76, p. 204)

Use logarithmic differentiation to find the first derivative of the functions

$$y = (\ln x)^{3x}$$

$$y = x^{\cos x}$$

$$y = (\cos x)^x$$



(a)  $y = (\ln x)^{3x}$   
 Take  $\ln y$  of both sides:  $\ln y = \ln[(\ln x)^{3x}]$   
 So  $\ln y = 3x \cdot \ln[\ln(x)]$ . Take  $\frac{d}{dx}$  of both sides:

$$\frac{d}{dx} [\ln y] = \frac{d}{dx} [3x \cdot \ln[\ln(x)]]$$

$$\frac{1}{y} \frac{dy}{dx} = 3 \cdot \ln(\ln(x)) + 3x \cdot \left(\frac{1}{\ln x} \cdot \frac{1}{x}\right)$$

$$\therefore \frac{dy}{dx} = y \left[ 3 \ln(\ln(x)) + \frac{3}{\ln x} \right]$$

$$= (\ln x)^{3x} \cdot \left[ 3 \ln(\ln(x)) + \frac{3}{\ln x} \right]$$

(b) Consider  $y = x^{\cos x}$ . Take:  $\ln y = \ln[x^{\cos x}]$

$$\Leftrightarrow \ln y = \cos x \cdot \ln x$$

Take  $\frac{d}{dx}$  of both sides:  $\frac{d}{dx} [\ln y] = \frac{d}{dx} [\cos x \cdot \ln x]$

$$\frac{1}{y} \frac{dy}{dx} = -\sin x \cdot \ln x + \cos x \cdot \frac{1}{x}$$

chain rule  $\therefore \frac{dy}{dx} = y \cdot \left[ -\sin x \ln x + \frac{\cos x}{x} \right]$

OR  $\frac{dy}{dx} = x^{\cos x} \cdot \left[ -\sin x \ln x + \frac{\cos x}{x} \right]$

(c)  $y = (\cos x)^x \Leftrightarrow \ln y = x \cdot \ln(\cos x)$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = 1 \cdot \ln(\cos x) + x \cdot \frac{1}{\cos x} (-\sin x)$$

$$\Rightarrow \frac{dy}{dx} = (\cos x)^x \cdot [\ln(\cos x) - x \tan x]$$

**Example 6:** (Neuhauser, Problem # 77, p. 204)

Use logarithmic differentiation to find the first derivative of the function

$$y = \frac{e^{2x}(9x-2)^3}{\sqrt[4]{(x^2+1)(3x^3-7)}}$$

$$y = \frac{e^{2x} \cdot (9x-2)^3}{\sqrt[4]{(x^2+1)(3x^3-7)}}$$

If we try to do the derivative with the quotient rule, it will be complicated.

Let's take "ln" of both sides:

$$\Leftrightarrow \ln y = \ln \left[ \frac{e^{2x}(9x-2)^3}{\sqrt[4]{(x^2+1)(3x^3-7)}} \right]$$

$$\Leftrightarrow \ln y = \ln [e^{2x}(9x-2)^3] - \ln [(x^2+1)(3x^3-7)]^{1/4}$$

$$\Leftrightarrow \ln y = \ln(e^{2x}) + \ln[(9x-2)^3] - \frac{1}{4} \left[ \ln[(x^2+1)(3x^3-7)] \right]$$

$$\Leftrightarrow \ln y = 2x + 3 \ln(9x-2) - \frac{1}{4} \ln(x^2+1) - \frac{1}{4} \ln(3x^3-7)$$



Hence, after expanding, we have that

$$y = \frac{e^{2x} \cdot (9x-2)^3}{4 \sqrt{(x^2+1)(3x^3-7)}} \quad (\Leftrightarrow)$$

$$\ln y = 2x + 3 \ln(9x-2) - \frac{1}{4} \ln(x^2+1) - \frac{1}{4} \ln(3x^3-7)$$

Take the derivative w.r.t  $x$  of both sides:

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2 + 3 \frac{1}{9x-2} \cdot (9) - \frac{1}{4} \frac{1}{x^2+1} \cdot (2x) - \frac{1}{4} \frac{1}{3x^3-7} \cdot (9x^2)$$

$$\text{hence } \frac{dy}{dx} = y \left[ 2 + \frac{27}{9x-2} - \frac{x}{2(x^2+1)} - \frac{9x^2}{4(3x^3-7)} \right]$$

$$\text{or } \frac{dy}{dx} = \frac{e^{2x} (9x-2)^3}{4 \sqrt{(x^2+1)(3x^3-7)}} \cdot \left[ 2 + \frac{27}{9x-2} - \frac{x}{2(x^2+1)} - \frac{9x^2}{4(3x^3-7)} \right]$$

## Power Rule (General Form)

### Theorem

Let  $f(x) = x^r$ , where  $r$  is any real number. Then

$$\frac{d}{dx} x^r = r x^{r-1}$$

**Proof:** We set  $y = x^r$  and use logarithmic differentiation to obtain

$$\frac{d}{dx} [\ln y] = \frac{d}{dx} [\ln x^r]$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{d}{dx} [r \ln x]$$

$$\frac{1}{y} \frac{dy}{dx} = r \frac{1}{x}$$

Solving for  $dy/dx$  yields

$$\frac{dy}{dx} = r \frac{1}{x} y = r \frac{1}{x} x^r = r x^{r-1}$$