MA 137 — Calculus 1 with Life Science Applications
Linear Approximations
(Section 4.11)

Department of Mathematics
University of Kentucky

Tangent Line Approximation

Assume that $y = f(x)$ is differentiable at $x = a$; then

$$L(x) = f(a) + f'(a)(x - a)$$

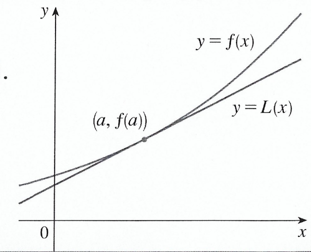
is the tangent line approximation, or **linearization**, of f at $x = a$.

Geometrically, the linearization of f at $x = a$ is the equation of the tangent line to the graph of $f(x)$ at the point $(a, f(a))$.

If $|x - a|$ is sufficiently small, then $f(x)$ can be linearly approximated by $L(x)$; that is,

$$f(x) \approx f(a) + f'(a)(x - a).$$

This approximation is illustrated in the picture on the right:



Example 1: (Nuehauser, Example # 1, p. 206)

(a) Find the linear approximation of $f(x) = \sqrt{x}$ at $x = a$.

(b) use your answer in (a) to find an approximate value of $\sqrt{26}$.

(a) $f(x) = \sqrt{x}$ and $f'(x) = \frac{1}{2\sqrt{x}}$

Thus the linearization of f at $x = a$ is

$$L(x) = f(a) + f'(a)\cdot(x-a)$$

$$\boxed{L(x) = \sqrt{a} + \frac{1}{2\sqrt{a}}(x-a)}$$

(b) In order to approximate $\sqrt{26}$ notice that 26 is close to 25; and we know that $\sqrt{25} = 5$

Thus we choose to linearize $f(x) = \sqrt{x}$ at $\boxed{a = 25}$

$$L(x) = 5 + \frac{1}{10}(x - 25)$$

Thus $\sqrt{26} \approx L(26) = 5 + \frac{1}{10}\underbrace{(26-25)}_{=1} = \boxed{5.1}$

Example 2: (Online Homework HW16, # 18)

Find the linearization $L(x)$ of the function $g(x) = x\,f(x^2)$ at $x = 2$ given the following information:

$$f(2) = 1 \qquad f'(2) = 10 \qquad f(4) = 5 \qquad f'(4) = -2$$

We want the linearization of $g(x) = x\,f(x^2)$ at $x=2$. Thus we need

$g(2) = 2\cdot f(2^2) = 2\cdot f(4) = 2\cdot 5 = \underline{10}$

and $g'(2)$. For this we need first $g'(x)$:

$$g'(x) = 1\cdot f(x^2) + x\cdot \underbrace{f'(x^2)\cdot 2x}_{\text{chain rule}}$$

$$= f(x^2) + 2x^2\cdot f'(x^2)$$

So $g'(2) = f(4) + 8\cdot f'(4) = 5 + 8(-2) = \underline{\underline{-11}}$

Thus: $L(x) = 10 - 11(x-2)$

$$\overset{!}{=} \underline{\underline{-11x + 32}}$$

Example 3: (Nuehauser, Problem # 34, p. 210)

Plant Biomass: Suppose that a certain plant is grown along a gradient ranging from nitrogen-poor to nitrogen-rich soil.

Experimental data show that the average mass per plant grown in a soil with a total nitrogen content of 1000 mg nitrogen per kg of soil is 2.7 g and the rate of change of the average mass per plant at this nitrogen level is 1.05×10^{-3} g per mg change in total nitrogen per kg soil.

Use a linear approximation to predict the average mass per plant grown in a soil with a total nitrogen content of 1100 mg nitrogen per kg of soil.

We know that $B(1{,}000) = 2.7$ and $\frac{dB}{dt}\Big|$

and $B'(1{,}000) = 1.05\times10^{-3}$

Thus the linearization is

$$L(x) = 2.7 + 1.05\times10^{-3}(x - 1{,}000)$$

Hence the value at $x = 1{,}100$ is

$$B(1{,}100) \approx L(1{,}100) = 2.7 + 1.05\times10^{-3}(1{,}100 - 1{,}000)$$

$$= 2.7 + 1.05\times10^{-3}(100)$$

$$= 2.7 + 0.105 = \underline{\underline{2.805}}$$

Example 4: (Nuehauser, Example # 3, p. 206)

Suppose $N = N(t)$ represents a population size at time t and the rate of growth as a function of N is $g(N)$.

Find the linear approximation of the growth rate at $N = 0$.

[**Hint:** We can assume that $g(0) = 0$. Indeed, when the population has size $N = 0$, its growth rate will be zero.]

[**Remark:** Your answer should show that for small population sizes, the population grows approximately exponentially.]

$\frac{dN}{dt}$ = growth rate $= g(N)$

We want to find the linearization of the growth rate at $N=0$:

$$L(N) = g(0) + g'(0)\cdot(N-0) = g(0) + g'(0)\cdot N$$

By assumption $g(0) = 0$; so $\underline{L(N) = g'(0)N}$

If we set $g'(0) = r$ then

$$\frac{dN}{dt} \approx L(N) = rN$$

, which describes an exp. growth

Example 5: (Neuhauser, Problem # 33, p. 210)

Plant Biomass: Suppose that the specific growth rate of a plant is 1%; that is, if $B(t)$ denotes the biomass at time t, then

$$\frac{1}{B(t)}\frac{dB}{dt} = 0.01$$

Suppose that the biomass at time $t = 1$ is equal to 5 grams.

Use a linear approximation to compute the biomass at time $t = 1.1$.

We know that $B(1) = 5$

and that $\frac{1}{B(t)}\frac{dB}{dt} = 0.01$

so $\left.\frac{dB}{dt}\right|_{t=1} = 0.01 \cdot B(1) = 0.01 \cdot 5 = 0.05$

Thus the linearization of the biomass at $t=1$ is:

$$L(t) = B(1) + \left.\frac{dB}{dt}\right|_{t=1}\cdot(t-1) = 5 + 0.05(t-1)$$

Hence $B(1.1) \cong L(1.1) = 5 + 0.05\,(1.1-1)$

$$= 5 + 0.05\,(0.1) = \boxed{5.005}$$

Higher Order Approximations

The tangent linear approximation $L(x) = f(a) + f'(a)(x-a)$ is the best first-degree (linear) approximation to $f(x)$ near $x = a$ because $f(x)$ and $L(x)$ have the same value and the same rate of change at a

$$L(a) = f(a) \qquad L'(a) = f'(a).$$

For a better approximation than a linear one, let's try to find better approximations by looking for an nth-degree polynomial

$$T_n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$$

such that T_n and its first n derivatives have the same value at $x = a$ as f and its first n derivatives at $x = a$.
We can show that the resulting polynomial is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

It is called the **nth-degree Taylor polynomial** of f centered at $x = a$.

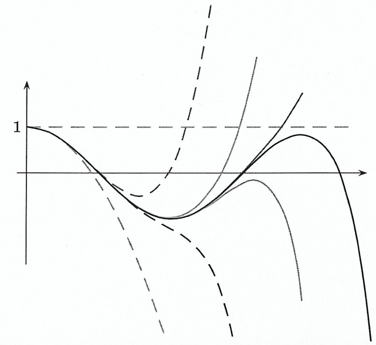
Linear Approximations | Theory | Examples | (Optional: Taylor Polynomials)

Approximation of $\cos x$ centered at $a = 0$

Consider the graph of the polynomial

$$T_{2n}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^{n-1}\frac{x^{2(n-1)}}{(2n-2)!} + (-1)^n\frac{x^{2n}}{(2n)!}.$$

As n increases, the graph of $T_{2n}(x)$ appears to approach the one of $\cos x$.
This suggests that we can approximate $\cos x$ with $T_{2n}(x)$ as $n \to \infty$.



$y = 1$
$y = 1 - \frac{x^2}{2!}$
$y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$
$y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$
$y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$
$y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}$
$y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!}$
$y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \frac{x^{14}}{14!}$

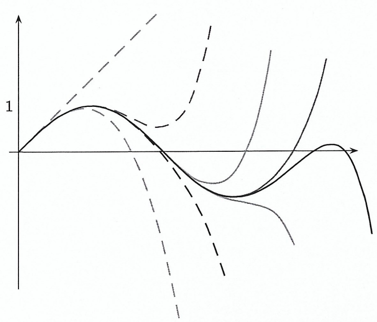
Linear Approximations | Theory | Examples | (Optional: Taylor Polynomials)

Approximation of $\sin x$ centered at $a = 0$

Consider the graph of the polynomial

$$T_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + (-1)^n\frac{x^{2n+1}}{(2n+1)!}.$$

As n increases, the graph of $T_{2n+1}(x)$ appears to approach the one of $\sin x$. This suggests that we can approximate $\sin x$ with $T_{2n+1}(x)$ as $n \to \infty$.



$y = x$
$y = x - \frac{x^3}{3!}$
$y = x - \frac{x^3}{3!} + \frac{x^5}{5!}$
$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$
$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$
$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$
$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!}$
$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!}$

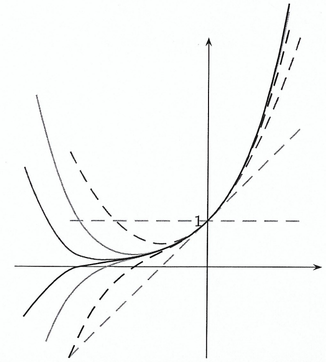
Linear Approximations | Theory | Examples | (Optional: Taylor Polynomials)

Approximation of e^x centered at $a = 0$

Consider the graph of the polynomial

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}.$$

As n increases, the graph of $T_n(x)$ appears to approach the one of e^x.
This suggests that we can approximate e^x with $T_n(x)$ as $n \to \infty$.



$y = 1$
$y = 1 + x$
$y = 1 + x + \frac{x^2}{2!}$
$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$
$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$
$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$
$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}$
$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}$

Example 6: (Bloomberg Business, 10/23/15)

Google parent Alphabet Inc. reached a record share price a day after reporting better-than-projected quarterly revenue and profit fueled by increased ad sales and a tighter lid on costs. [...] The actual figure that the company announced for the share buyback was unusually specific: $5,099,019,513.59. Turns out, those numbers correspond to the square root of 26, or the number of letters in the English alphabet.

— ... —

Let $f(x) = \sqrt{x}$ and $a = 25$. The 5th-degree Taylor polynomial of f centered at 25 can be shown to be

$$T_5(x) = 5 + \frac{1}{10}(x-25) - \frac{1}{1,000}(x-25)^2 + \frac{1}{50,000}(x-25)^3 - \frac{1}{2,000,000}(x-25)^4 + \frac{1}{71,428,571.43}(x-25)^5$$

We can then check that

$$\sqrt{26} \approx T_5(26) = 5 + \frac{1}{10} - \frac{1}{1,000} + \frac{1}{50,000} - \frac{1}{2,000,000} + \frac{1}{71,428,571.43} = 5.099019514$$

This means that we overestimated Alphabet Inc. buyback by 41¢.