# MA 137 — Calculus 1 with Life Science Applications
# **Extrema and The Mean Value Theorem**
# (Section 5.1)

Department of Mathematics
University of Kentucky





Finding the largest profit, or the smallest possible cost, or the shortest possible time for performing a given procedure or task are some examples of practical real-world applications of Calculus.

The basic mathematical question underlying such applied problems is how to find (if they exist) the largest or smallest values of a given function on a given interval.

This procedure depends on the nature of the interval.

## Global (or Absolute) Extreme Values

The largest value a function (possibly) attains on an interval is called its **global (or absolute) maximum value**.

The smallest value a function (possibly) attains on an interval is called its **global (or absolute) minimum value**.

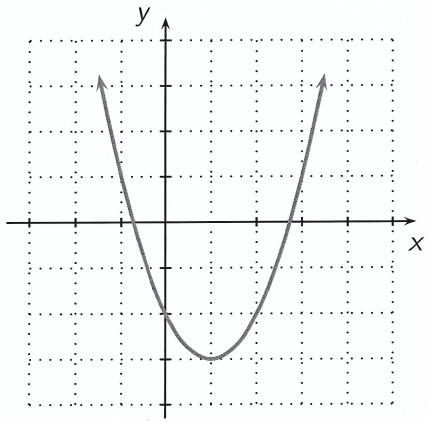
Both maximum and minimum values (if they exist) are called **global (or absolute) extreme values**.





## Example 1:

Find the maximum and minimum values for the function $f(x) = (x-1)^2 - 3$, if they exist.





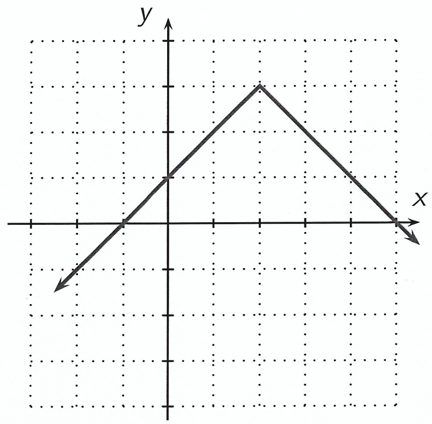

The function has no global maximum since it grows without any bound.

It has a global minimum of -3, which is attained when $x = 1$. It corresponds to the vertex of the parabola.

(Notice that the vertex corresponds to the point where $f'(x) = 0$)

## Example 2:

Find the maximum and minimum values for the function $f(x) = -|x-2| + 3$, if they exist.






The function has no global minimum.

It has a global maximum of 3, which is attained when $x = 2$.

(notice that the global max is attained at the point where $f'(x)$ does not exist)

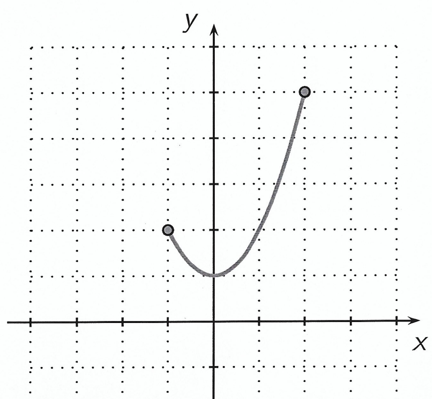


## Example 3:

Find the maximum and minimum values for the function

$$f(x) = x^2 + 1, \qquad x \in [-1, 2]$$

if they exist.






The function has a global maximum of 5, which is attained when $x = 2$.

The function has a global minimum of 1, which is attained when $x = 0$.

## The Extreme Value Theorem (EVT)

We first focus on continuous functions on a closed and bounded interval.

The question of largest and smallest values of a continuous function $f$ on an interval that is not closed and bounded requires us to pay more attention to the behavior of the graph of $f$, and specifically to where the graph is rising and where it is falling.

**Closed and bounded intervals**

An interval is **closed and bounded** if it has finite length and contains its endpoints.

For example, the interval $[-2, 5]$ is closed and bounded.

**Theorem (The Extreme Value Theorem)**

*If a function $f$ is continuous on a closed, bounded interval $[a, b]$, then the function $f$ attains a global maximum and a global minimum value on $[a, b]$.*

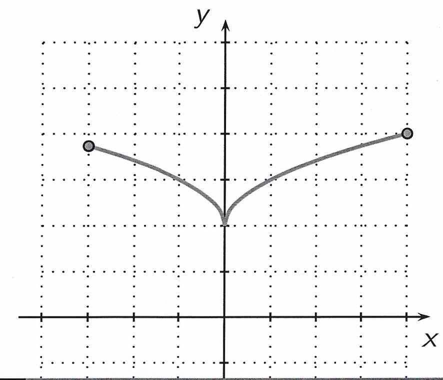




## Example 4:

Let $f(x) = \begin{cases} 2 + \sqrt{x} & \text{if } x > 0 \\ 2 + \sqrt{-x} & \text{if } x \leq 0. \end{cases}$

Does $f(x)$ have a maximum and a minimum value on $[-3, 4]$? How does this example illustrate the Extreme Value Theorem?






Yes the function is continuous on a closed interval hence by the Extreme Value Theorem it has a global max. and a global minimum.

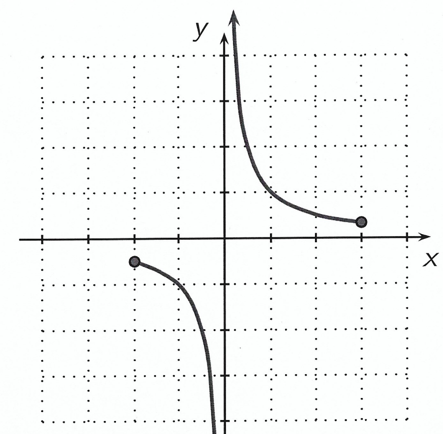
The global max is 4, which is attained when $x = 4$.

The global min is 2, which is attained when $x = 0$.



## Example 5:

Let $g(x) = \dfrac{1}{x}$. Does $g(x)$ have a maximum value and a minimum value on $[-2, 3]$? Does this example contradict the Extreme Value Theorem? Why or why not?



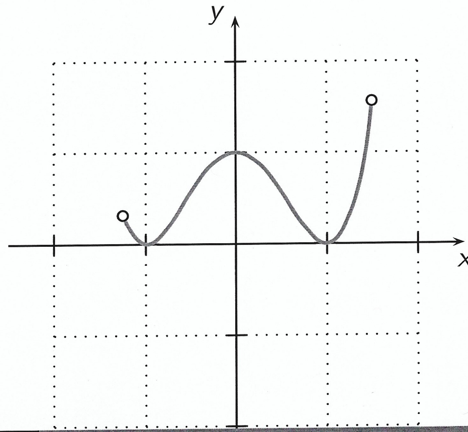
The function is not continuous on the closed interval $[-2, 3]$. Hence the Extreme Value Theorem does not apply.

The function has no global max nor a global minimum.



## Example 6:

Let $h(x) = x^4 - 2x^2 + 1$. Does $h(x)$ have a maximum value and a minimum value on $(-1.25, 1.5)$? Does this example contradict the Extreme Value Theorem? Why or why not?






The function $h(x)$ has a global minimum on the interval $(-1.25, 1.5)$ but has no global maximum.

In fact, the function $h(x)$ is continuous but the interval over which we are considering it is not closed. Hence the EVT does not apply.



## Local Extreme Points

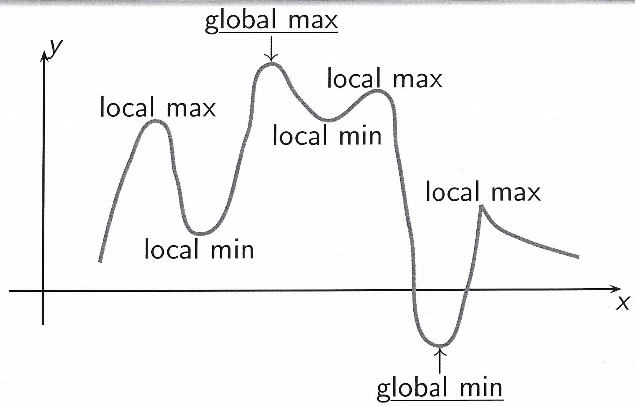
The EVT is an existence statement; it doesn't tell you how to locate the maximum and minimum values of $f$.

We need to narrow down the list of possible points on the given interval where the function $f$ *might* have an extreme value to (usually) just a few possibilities. You can then evaluate $f$ at these few possibilities, and pick out the smallest and largest value.

For this we need to discuss local (or relative) extrema, which are points where a graph is higher or lower than all *nearby* points.

### Local (or relative) extreme points

A function $f$ has a **local** (or **relative**) **maximum** at a point $(c, f(c))$ if there is some interval about $c$ such that $f(c) \geq f(x)$ for all $x$ in that interval. A function $f$ has a **local** (or **relative**) **minimum** at a point $(c, f(c))$ if there is some interval about $c$ such that $f(c) \leq f(x)$ for all $x$ in that interval.




If you thought of the graph of the function as the profile of a landscape, the global maximum could represent the highest hill in the landscape, while the minimum could represent the deepest valley. The other points indicated in the graph, which look like tops of hills (although not the highest hills) and bottom of valleys (although not the deepest valleys), are the **local** (or **relative**) **extreme values**.





## Fermat's Theorem

**Theorem (Fermat's Theorem)**

*Let $f(x)$ be a continuous function. If $f$ has an extreme value at an interior point $c$ and if $f$ is differentiable at $x = c$, then $f'(c) = 0$.*

This results provide the following guidelines for finding candidates for local extrema:

**Corollary**

*Let $f(x)$ be a continuous function on the closed, bounded interval $[a, b]$. If $f$ has an extreme value at $c$ in the interval, then either*

- $c = a$ *or* $c = b$;
- $a < c < b$ *and* $f'(c) = 0$;
- $a < c < b$ *and* $f'$ *is not defined at* $x = c$.

**Remark:** If $f$ is defined at the point $x = c$ and either $f'(c) = 0$ or $f'(c)$ is undefined then the point $c$ is called a **critical point** of $f$.





## Example 7:

Find the maximum and minimum values of
$f(x) = x^3 - 3x^2 - 9x + 5$ on the interval $[0, 4]$. For which values $x$ are the maximum and minimum values attained?



The function $f(x) = x^3 - 3x^2 - 9x + 5$ is continuous for every $x \in \mathbb{R}$. Hence it is continuous on $[0,4]$. Moreover $[0,4]$ is a closed and bounded interval. Hence by the EVT it has a global max and a global min. We need to check the value of $f$ at the end point and when $f'(x)=0$ (since $f'$ exists everywhere).

$f'(x) = 3x^2 - 6x - 9 = 0 \iff x^2 - 2x - 3 = 0$

$\iff (x-3)(x+1) = 0$. $\therefore x = 3, -1$

Notice that only $3 \in [0,4]$.

| | $x$ | $f(x)$ |
|---|---|---|
| endpoints | $0$ | $f(0)=5$ |
| | $4$ | $f(4)=-15$ |
| critical # | $3$ | $f(3)=-22$ |

Hence $f$ has a global maximum of $5$ attained at $x=0$. $f$ has a global minimum of $-22$ attained at $x=3$.



## Example 8: (Online Homework HW17, # 7)

Find the maximum and minimum values of $f(x) = \dfrac{4x}{x^2+1}$ on the interval $[-4, 0]$. For which values $x$ are the maximum and minimum values attained?



The function $f(x) = \frac{4x}{x^2+1}$ is continuous for every $x \in \mathbb{R}$ as the denominator is never $0$, and it is the quotient of two continuous functions. In particular it is continuous on the closed interval $[-4,0]$. Hence by the EVT it has a global max and a global minimum.

$$f'(x) = \frac{4(x^2+1) - 4x(2x)}{(x^2+1)^2} = \frac{4x^2+4-8x^2}{(x^2+1)^2} = \frac{4-4x^2}{(x^2+1)^2}$$

$f'$ is defined for every $x$. $f'(x)=0$

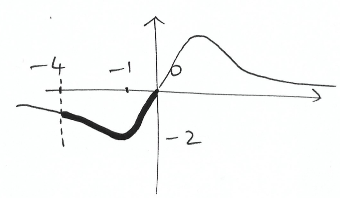
$\iff 4-4x^2=0 \iff x^2=1 \iff x=\pm 1$

Notice that only $-1 \in [-4,0]$.

| | $x$ | $f(x)$ |
|---|---|---|
| endpoints of interval | $-4$ | $f(-4) = -16/17$ |
| | $0$ | $f(0) = 0$ |
| critical # | $-1$ | $f(-1) = -2$ |

Hence $f$ has a global maximum of $0$, attained at $x=0$.

$f$ has a global minimum of $-2$, attained at $x=-1$.



## Example 9:

Find the maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-1, 8]$. For which values $x$ are the maximum and minimum values attained?



The function is continuous for every $x$; in particular it is continuous on the closed interval $[-1, 8]$. Thus by the EVT it has a global max and a global min.

Notice that $f'(x) = \frac{2}{3} x^{2/3 - 1} = \frac{2}{3} x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$

Thus $f'(x) = 0$ cannot occur.

However $f'$ is not differentiable at $x = 0$.

| | end points | | critical # |
|---|---|---|---|
| $x$ | $-1$ | $8$ | $0$ |
| $f(x)$ | $1$ | $4$ | $0$ |

- global max 4 attained at $x = 8$
- global min 0 attained at $x = 0$



## Example 10:

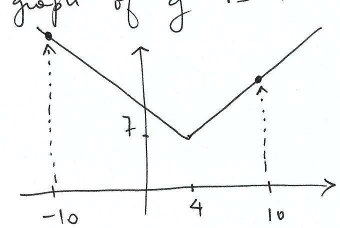
Find the $t$ values on the interval $[-10, 10]$ where $g(t) = |t - 4| + 7$ takes its maximum and minimum values. What are the maximum and minimum values?



The function $g(t)$ is continuous everywhere; in particular it is continuous on the closed interval $[-10, 10]$. Hence by the EVT it has a global max and a global min.

We need to test the value of $g$ at the end points of the intervals and where $g' = 0$ or does not exist. Notice that $g'$ is never $0$, but it does not exist at $t = 4$.

The graph of $g$ is:




| | $t$ | $g(t)$ | |
|---|---|---|---|
| end points | $-10$ | $21$ | ← global max |
| | $10$ | $13$ | |
| critical # | $4$ | $7$ | ← global min |

## Example 11: (Online Homework HW17, # 9)

Find the absolute maximum and minimum values of the function

$$f(x) = \frac{10\cos x}{4 + 2\sin x}$$

over the interval $[0, 2\pi]$. If there are multiple points in a single category list the points in increasing order in $x$ value.



$$f(x) = \frac{10\cos x}{4 + 2\sin x}$$

is a continuous function over the closed interval $[0, 2\pi]$. Hence by the EVT it has a global max and a global minimum.

We need to find the values where $f'(x) = 0$.
Notice that $f$ is differentiable everywhere:

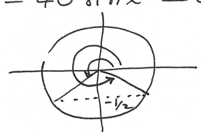
$$f'(x) = \frac{10(-\sin x)(4 + 2\sin x) - 10\cos x\,(+2\cos x)}{(4 + 2\sin x)^2}$$

$$= \frac{-40\sin x - 20\sin^2 x - 20\cos^2 x}{(4 + 2\sin x)^2}$$

$$= \frac{-20 - 40\sin x}{(4 + 2\sin x)^2} \quad \text{as } \underline{\sin^2 x + \cos^2 x = 1}$$

Thus $f'(x) = 0 \iff -20 - 40\sin x = 0$

$\iff \sin x = -\frac{1}{2}$.




Notice that this occurs for

$x = \pi + \frac{\pi}{6} = \frac{7}{6}\pi$ and $x = 2\pi - \pi/6 = \frac{11}{6}\pi$.

Hence:

| | $x$ | $f(x)$ |
|---|---|---|
| endpoints | $0$ | $5/2 = 2.5$ |
| | $2\pi$ | $5/2 = 2.5$ |
| critical #s | $\frac{7}{6}\pi$ | $f(\frac{7}{6}\pi) = \frac{10\cos(\frac{7}{6}\pi)}{4 + 2\sin(\frac{7}{6}\pi)} = \frac{10(-\sqrt{3}/2)}{4 + 2(-1/2)} = \frac{-5\sqrt{3}}{3} \cong -2.8868$ |
| | $\frac{11}{6}\pi$ | $f(\frac{11}{6}\pi) = \frac{10(+\sqrt{3}/2)}{4 + 2(-1/2)} = \frac{5}{3}\sqrt{3} \cong 2.8868$ |

Hence the global max is 2.8868 at $x = \frac{11}{6}\pi$ and the global min is $-2.8868$ at $x = \frac{7}{6}\pi$