MA 137 — Calculus 1 with Life Science Applications
Extrema and The Mean Value Theorem
(Section 5.1)

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The Mean Value Theorem (MVT)

The Mean Value Theorem is a very important in calculus. Its consequences are far reaching, and we will use it to derive important results that will help us to analyze functions.

Theorem (Mean Value Theorem)

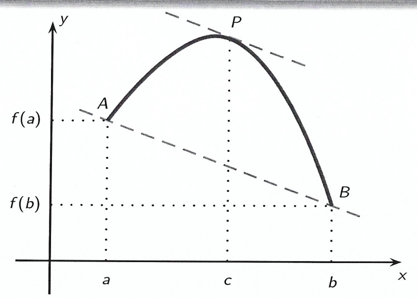
If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists at least one number $c \in (a, b)$ such that

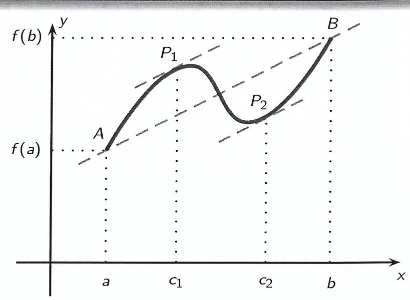
$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Geometrically, it says that there exists a point $P(c, f(c))$ on the graph where the tangent line at this point is parallel to the secant line through $A(a, f(a))$ and $B(b, f(b))$.

The MVT is an "existence" result: It tells us neither how many such points there are nor where they are in the interval (a, b).

Geometric Interpretation and a Special Case





The proof of the MVT is typically done by first showing a special case of the theorem called Rolle's Theorem.
You can read its proof on p. 211 of the Neuhauser book.

Theorem (Rolle's Theorem – 1691)

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and if $f(a) = f(b)$, then there exists a number $c \in (a, b)$ such that $f'(c) = 0$.

The MVT follows from Rolle's theorem and is a "tilted" version of that theorem. The secant and tangent lines in the MVT are no longer necessarily horizontal, as in Rolle's theorem, but are "tilted"; they are still parallel, though.

Proof of the MVT: We define the following function:

$$F(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

The function F is continuous on $[a, b]$ and differentiable on (a, b). Furthermore, $F(a) = f(a) = F(b)$. Hence, we can apply Rolle's theorem to the function $F(x)$. There exists a $c \in (a, b)$ with $F'(c) = 0$. Since

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

it follows that, for this value of c,

$$0 = F'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

and hence

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example 1: (Online Homework HW17, # 10)

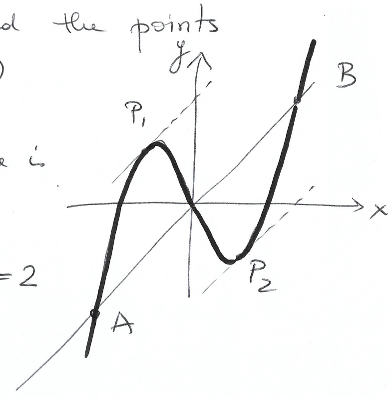
Graph the function $f(x) = x^3 - 2x$ and its secant line through the points $(-2,-4)$ and $(2,4)$. Use the graph to estimate the x-coordinate of the points where the tangent line is parallel to the secant line.

Find the exact value of the numbers c that satisfy the conclusion of the Mean Value Theorem for the interval $[-2,2]$.

Consider $f(x) = x^3 - 2x$ and the points $A(-2,-4)$ and $B(2,4)$

the slope of the secant line is

$$\frac{f(b)-f(a)}{b-a} = \frac{4-(-4)}{2-(-2)} = \frac{8}{4} = 2$$



Now, $f'(x) = 3x^2 - 2$

To find $(c, f(c))$ as in the Mean Value Theorem we need to solve:

$$f'(c) = 2 \iff 3x^2 - 2 = 2 \iff x^2 = 4/3$$

$$\iff \boxed{x = \pm \tfrac{2}{3}\sqrt{3}}$$

$\therefore$ $P_1(-1.1547, 0.7698)$
$P_2(1.1547, -0.7698)$

Example 2: (Online Homework HW17, # 12)

Find all numbers c that satisfy the conclusion of Rolle's Theorem for the following function

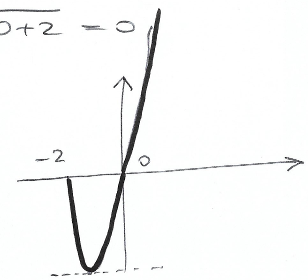
$$f(x) = 9x\sqrt{x+2}$$

on the interval $[-2,0]$.

$f(x) = 9x\sqrt{x+2}$ on $[-2,0]$ is continuous on $[-2,0]$ and differentiable on $(-2,0)$

Notice that $f(-2) = 9(-2)\sqrt{-2+2} = 0$
$f(0) = 9\cdot 0\sqrt{0+2} = 0$

We want to find c in $[-2,0]$ such that $f'(c) = 0$.



$$f'(x) = 9\cdot 1\cdot\sqrt{x+2} + 9\cdot x\cdot \underbrace{\frac{1}{2\sqrt{x+2}}\cdot 1}_{\text{chain rule}}$$

$$f'(x) = \frac{18(\sqrt{x+2})^2 + 9x}{2\sqrt{x+2}} = \frac{27x+36}{2\sqrt{x+2}}$$

not differentiable at $x=-2$

Hence $f'(c) = 0 \iff \dfrac{27c+36}{2\sqrt{c+2}} = 0 \iff$

$27c + 36 = 0$ $\boxed{c = -\frac{36}{27} \simeq -1.334}$

Example 3: (Online Homework HW17, # 13)

Consider the function $f(x) = 3 - 3x^{2/3}$ on the interval $[-1, 1]$. Which of the three hypotheses of Rolle's Theorem fails for this function on the interval?

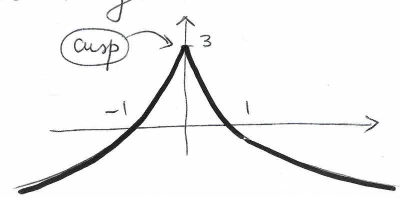
(a) $f(x)$ is continuous on $[-1, 1]$.
(b) $f(x)$ is differentiable on $(-1, 1)$.
(c) $f(-1) = f(1)$.

$f(x) = 3 - 3x^{2/3}$

* is continuous on $[-1, 1]$
* $f(-1) = f(1) = 0$
* <u>But</u> $f(x)$ is <u>not</u> differentiable at $x = 0$

in fact $f'(x) = -3 \cdot \frac{2}{3} x^{2/3 - 1} = -2x^{-1/3} \stackrel{!}{=} -\frac{2}{\sqrt[3]{x}}$

hence at $x=0$ the tangent line is vertical



Consequences of the MVT

We discuss two consequences of the MVT.

The first corollary is useful in obtaining information about a function on the basis of its derivative. The importance of the second corollary will become more apparent in Example 7 and Section 5.8.

Corollary 1

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) such that

$$m \le f'(x) \le M \qquad \text{for all } x \in (a, b)$$

then

$$m(b-a) \le f(b) - f(a) \le M(b-a)$$

Corollary 2

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), with $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.

Example 4: (Online Homework HW17, # 14)

Suppose $f(x)$ is continuous on $[3, 5]$ and

$$-5 \le f'(x) \le 2$$

for all x in $(3, 5)$.
Use the Mean Value Theorem to estimate $f(5) - f(3)$.

$f(x)$ is continuous on $[3,5]$ and

$$-5 \le f'(x) \le 2$$

for all $x \in (3,5)$. By the MVT there exists $c \in (3,5)$ such that $f'(c) = \dfrac{f(5)-f(3)}{5-3}$

Hence for that particular c:

$$-5 \le f'(c) \le 2$$

$$\Longleftrightarrow$$

$$-5 \le \frac{f(5)-f(3)}{2} \le 2$$

$$\Longleftrightarrow$$

$$\boxed{-10 \le f(5)-f(3) \le 4}$$

Example 5: (Neuhauser, Example # 8, p. 221)

Denote the population size at time t by $N(t)$, and assume that $N(t)$ is continuous on the interval $[0,10]$ and differentiable on the interval $(0,10)$ with $N(0) = 100$ and $\left|\dfrac{dN}{dt}\right| \le 3$ for all $t \in (0,10)$.

What can you say about $N(10)$?

We know that $N(t)$ is continuous on $[0,10]$ and differentiable on $(0,10)$.

Moreover $N(0)=100$ and $-3 \le N'(t) \le 3$ for all $t \in (0,10)$

By the MVT there exists $c \in (0,10)$ such that

$$N'(c) = \frac{N(10)-N(0)}{10-0}$$

For that c we have the estimate

$$-3 \le N'(c) \le 3$$

$$\Longleftrightarrow$$

$$-3 \le \frac{N(10)-N(0)}{10} \le 3$$

$$\Longleftrightarrow \quad -30 \le N(10)-N(0) \le 30 \quad \Longleftrightarrow$$

$$\boxed{N(0)-30 \le N(10) \le N(0)+30} \Longleftrightarrow \boxed{70 \le N(10) \le 130}$$

Example 6: (Online Homework HW17, # 15)

Let $f(x) = 8\sin(x)$.

(a) $|f'(x)| \le$ ____

(b) By the Mean Value Theorem,

$$|f(b)-f(a)| \le \_\_\_\_\ |a-b|$$

for all a and b.

[**Remark:** This problem is also a variation of Example 9, Neuhauser, p. 212]

Let $f(x) = 8\sin x$. Then $f'(x) = 8\cos x$.

Since $-1 \le \cos x \le 1$ for all x, then

$$-8 \le f'(x) = 8\cos x \le 8$$

for all x. Or $|f'(x)| \le 8$.

In particular this is true for all $x \in [a,b]$.

By the MVT there is a $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Hence

$$-8 \le f'(c) \le 8 \iff$$

$$-8 \le \frac{f(b)-f(a)}{b-a} \le 8$$

or

$$\left| \frac{f(b)-f(a)}{b-a} \right| \le 8 \quad \text{or} \quad \boxed{|f(b)-f(a)| \le 8|x-a|}$$

Example 7: (Neuhauser, Problem # 56, p. 224)

We have seen that $f(x) = f_0 e^{rx}$ satisfies the differential equation $\frac{df}{dx} = r\,f(x)$ with $f(0) = f_0$.

This exercise will show that $f(x)$ is in fact the only solution.

Suppose that r is a constant and f is a differentiable function with

$$\frac{df}{dx} = r\,f(x) \tag{1}$$

for all $x \in \mathbb{R}$, and $f(0) = f_0$. The following steps will show that $f(x) = f_0 e^{rx}$, $x \in \mathbb{R}$, is the only solution of (1).

(a) Define the function $F(x) = f(x)e^{-rx}$, $x \in \mathbb{R}$. Use the product rule to show that $F'(x) = e^{-rx}[f'(x) - rf(x)]$.

(b) Use **(a)** and (1) to show that $F'(x) = 0$ for all $x \in \mathbb{R}$.

(c) Use Corollary 2 to show that $F(x)$ is a constant and, hence, $F(x) = F(0) = f_0$.

(d) Show that (c) implies that $f_0 = f(x)e^{-rx}$ and therefore, $f(x) = f_0 e^{rx}$.

Suppose that f is a solution of $\boxed{\frac{df}{dx} = rf}$ and satisfies $f(0) = f_0$.

(a) Define a new function $\boxed{F(x) = f(x) \cdot e^{-rx}}$

Then the derivative $F'(x)$ is:

$$F'(x) = f'(x)e^{-rx} + f(x) \cdot \underbrace{e^{-rx} \cdot (-r)}_{\text{chain rule}}$$

$$= e^{-rx} \cdot \left(f'(x) - rf(x) \right)$$

(b) But $\frac{df}{dx} = rf \iff f'(x) - rf(x) = 0$

Hence $F'(x) = e^{-rx} \cdot [\,0\,] = 0$

for all $x \in \mathbb{R}$.

(c) Since $F(x)$ is continuous for all x in any closed interval and differentiable for all x in the same open interval, with $F'(x) = 0$.

Then by Corollary 2, $F(x)$ is constant.

$$\boxed{F(x) = f(x)e^{-rx} = \text{Constant}}$$

Evaluate it at 0: $F(0) = f_0 \cdot e^{-r \cdot 0} = f_0$
$= \text{constant}$

(d) Thus $\boxed{F(x) = f(x)e^{-rx} = f_0}$

or $\boxed{f(x) = f_0 e^{rx}}$