MA 137 — Calculus 1 with Life Science Applications
Monotonicity and Concavity
(Section 5.2)
Extrema, Inflection Points, and Graphing
(Section 5.3 and 5.6)

Department of Mathematics
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Increasing and Decreasing Functions

A function f is said to be increasing when its graph rises and decreasing when its graph falls. More precisely, we say that

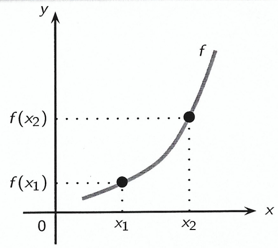
Definition

f is **(strictly) increasing** on an interval I if

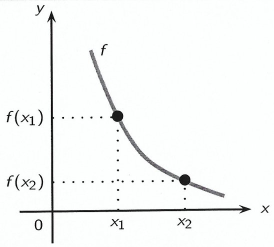
$$f(x_1) < f(x_2) \quad \text{whenever} \quad x_1 < x_2 \text{ in } I$$

f is **(strictly) decreasing** on an interval I if

$$f(x_1) > f(x_2) \quad \text{whenever} \quad x_1 < x_2 \text{ in } I$$



f is increasing



f is decreasing

First Derivative Test for Monotonicity

Theorem (First Derivative Test for Monotonicity)

Suppose f is continuous on $[a, b]$ and differentiable on (a, b).

(a) *If $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$.*

(b) *If $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$.*

Proof: Suppose $f'(x) > 0$ on an interval I. We wish to show that $f(x_1) < f(x_2)$ for any pair $x_1 < x_2$ in $[a, b]$.

Let x_1 and x_2 be any pair of point in $[a, b]$ satisfying $x_1 < x_2$. Then f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). We can therefore apply the MVT to f defined on $[x_1, x_2]$: There exists a number $c \in (x_1, x_2)$ such that

$$\frac{f(x_1) - f(x_2)}{x_2 - x_1} = f'(c)$$

Now, $f'(c) > 0$ as $c \in [x_1, x_2] \subset [a, b]$; so

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$$

so $f(x_2) - f(x_1) > 0$, since $x_2 - x_1 > 0$. Therefore, $f(x_1) < f(x_2)$.
Because x_1 and x_2 are arbitrary numbers in $[a, b]$ satisfying $x_1 < x_2$, it follows that f is increasing on the whole interval.

The proof of part (b) is similar.

First DerivativeTest for (Local) Extrema

Theorem (First Derivative Test for (Local) Extrema)

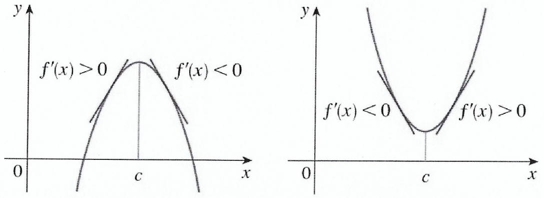
If f has a critical value at $x = c$, then

- *f has a local maximum at $x = c$ if the sign of f' around c is*

$$\begin{array}{c} +++ \quad --- \\ \hline c \end{array}$$

- *f has a local minimum at $x = c$ if the sign of f' around c is*

$$\begin{array}{c} --- \quad +++ \\ \hline c \end{array}$$

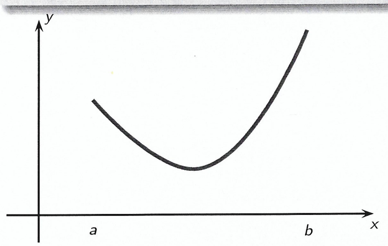


(a) Local maximum (b) Local minimum

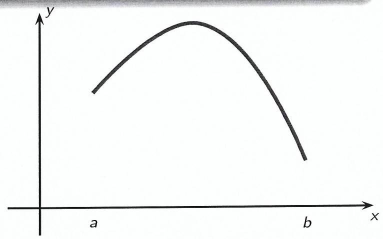
Concavity

The second derivative can also be used to help sketch the graph of a function. More precisely, the second derivative can be used to determine when the graph of a function is concave upward or concave downward.

The graph of a function $y = f(x)$ is **concave upward** on an interval $[a, b]$ if the graph lies above each of the tangent lines at every point in the interval $[a, b]$. The graph of a function $y = f(x)$ is **concave downward** on an interval $[a, b]$ if the graph lies below each of the tangent lines at every point in the interval $[a, b]$.



graph of function concave upward on $[a, b]$



graph of function concave downward on $[a, b]$

Second Derivative Test for Concavity

Consider a function $f(x)$.

If $f''(x) > 0$ over an interval $[a, b]$, then the derivative $f'(x)$ is increasing on the interval $[a, b]$. That means the slopes of the tangent lines to the graph of $y = f(x)$ are increasing on the interval $[a, b]$. From this it can be seen that the graph of the function $y = f(x)$ is concave upward.

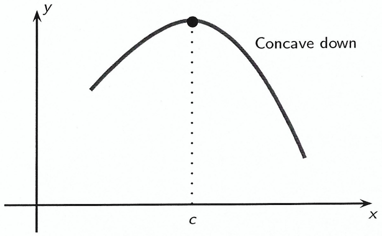
If $f''(x) < 0$ over an interval $[a, b]$. Then the derivative $f'(x)$ is decreasing on the interval $[a, b]$. That means the slopes of the tangent lines to the graph of $y = f(x)$ are decreasing on the interval $[a, b]$. From this it can be seen that the graph of the function $y = f(x)$ is concave downward.

Second DerivativeTest for (Local) Extrema

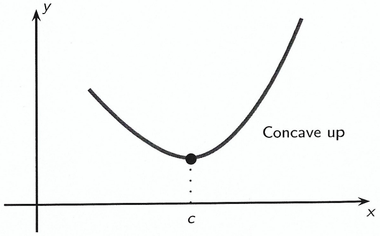
Theorem (Second Derivative Test for (Local) Extrema)

Suppose that f is twice differentiable on an open interval containing c.

- *If $f'(c) = 0$ and $f''(c) < 0$, then f has a local max. at $x = c$.*
- *If $f'(c) = 0$ and $f''(c) > 0$, then f has a local min. at $x = c$.*



f has a local max at c



f has a local min at c

Inflection Points

A point $(c, f(c))$ on the graph is called a **point of inflection** if the graph of $y = f(x)$ changes concavity at $x = c$. That is, if the graph goes from concave up to concave down, or from concave down to concave up.

If $(c, f(c))$ is a point of inflection on the graph of $y = f(x)$ and if the second derivative is defined at this point, then $f''(c) = 0$.

Thus, points of inflection on the graph of $y = f(x)$ are found where either $f''(x) = 0$ or the second derivative is not defined.

However, if either $f''(x) = 0$ or the second derivative is not defined at a point, it is not necessarily the case that the point is a point of inflection. Care must be taken.

About Graphing a Function

Using the first and the second derivatives of a twice-differentiable function, we can obtain a fair amount of information about the function.

We can determine intervals on which the function is increasing, decreasing, concave up, and concave down. We can identify local and global extrema and find inflection points.

To graph the function, we also need to know how the function behaves in the neighborhood of points where either the function or its derivative is not defined, and we need to know how the function behaves at the endpoints of its domain (or, if the function is defined for all $x \in \mathbb{R}$, how the function behaves for $x \to \pm\infty$).

A line $y = b$ is a horizontal asymptote if either

$$\lim_{x\to+\infty} f(x) = b \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = b$$

A line $x = c$ is a vertical asymptote if

$$\lim_{x\to c^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x\to c^-} f(x) = \pm\infty$$

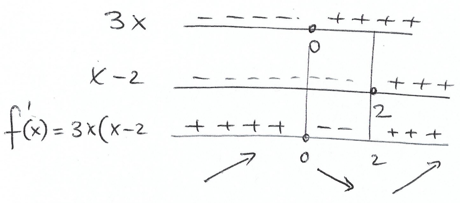
Example 1:

Find the intervals where the function $f(x) = x^3 - 3x^2 + 1$ is increasing and the ones where it is decreasing. Use this information to sketch the graph of $f(x) = x^3 - 3x^2 + 1$.

$f(x) = x^3 - 3x^2 + 1$

* We want to find the intervals when f is increasing and decreasing

$f'(x) = 3x^2 - 6x = 3x(x-2)$



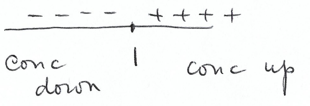
∴ f is increasing on $(-\infty, 0)$ and $(2, +\infty)$

∴ f is decreasing on $(0, 2)$

There is a local max at $x = 0$; a local min at $x = 2$

* About concavities: $f''(x) = 6x - 6 = 6(x-1)$

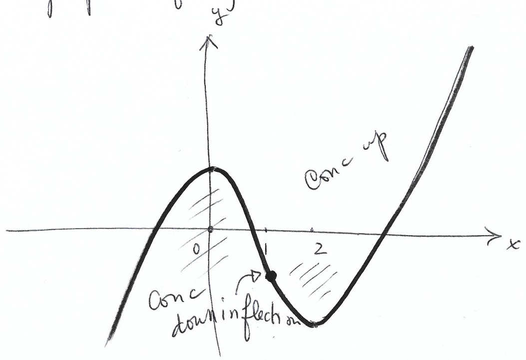
sign f''



Hence f is concave down on $(-\infty, 1)$ and it is concave up on $(1, +\infty)$

There is an inflection point at $x = 1$.

The graph of f looks like



	x	f(x)
local max	0	1
inflection	1	−1
local min →	2	−3

Note: it is hard to find where the graph meets precisely the x-axis!

Example 2:

Let $f(x) = \dfrac{x+4}{x+7}$. Find the intervals over which the function is increasing.

$f(x) = \dfrac{x+4}{x+7}$ not defined at $x = -7$

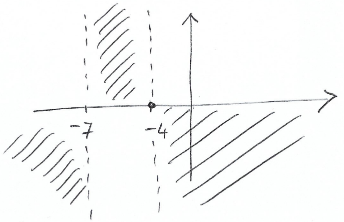
* Notice that the sign of $f(x)$ is as follows

$x+4$ ‒ ‒ ‒ ‒ $\underset{-4}{\circ}$ + + + + +

$x+7$ ‒ ‒ $\underset{-7}{\circ}$ + + + + + + + +

$f(x) = \dfrac{x+4}{x+7}$ + + $\underset{-7}{\otimes}$ ‒ ‒ ‒ $\underset{-4}{\circ}$ + + + + + +

hence the graph of f lies in :



* Let's find $f'(x)$.

$$f'(x) = \frac{1\cdot(x+7) - (x+4)(1)}{(x+7)^2} = \frac{\cancel{x}+7-\cancel{x}-4}{(x+7)^2} = \frac{3}{(x+7)^2}$$

The sign of f is always strictly positive ;
Notice f' is not differentiable at -7.

sign f' : + + + $\underset{-7}{\otimes}$ + + + +
$\nearrow$ $\nearrow$

Hence f' is always increasing

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* Let's also look at the concavities of f.

We need $f''(x)$.

$$f'(x) = \frac{3}{(x+7)^2} = 3(x+7)^{-2} \Longrightarrow f''(x) = -6(x+7)^{-3}\cdot(1)$$

$$\therefore \quad f'' = \frac{-6}{(x+7)^3}$$

-6 ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒ ‒

$(x+7)^3$ ‒ ‒ ‒ ‒ $\underset{-7}{\circ}$ + + + + + +

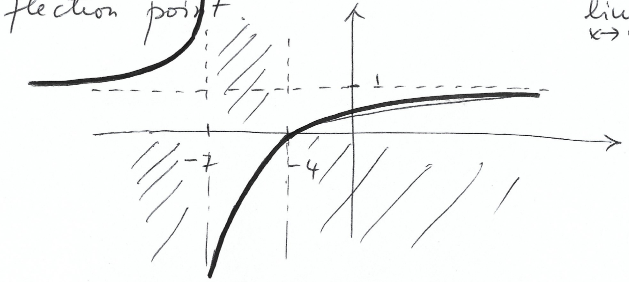
$f'' = \dfrac{-6}{(x+7)^3}$ + + + $\underset{-7}{\circ}$ ‒ ‒ ‒ ‒ ‒ ‒

Conc up conc down

Notice that there is <u>no</u> inflection point

$$\lim_{x\to\infty} \frac{x+4}{x+7} = 1$$

graph



Example 3:

Let $h(x) = x^2 e^{-x}$.

- **(a)** On what intervals is h increasing or decreasing?
- **(b)** At what values of x does h have a local maximum or minimum?
- **(c)** On what intervals is h concave upward or downward?
- **(d)** State the x-coordinate of the inflection point(s) of h.
- **(e)** Use the information in the above to sketch the graph of h.

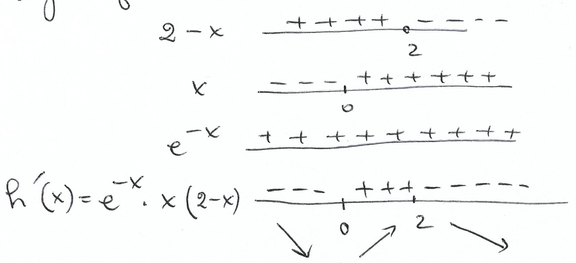
$h(x) = x^2 e^{-x}$

* Notice that $x^2 \geq 0$ for all x and $e^{-x} > 0$ for all x

Thus $h(x) = x^2 e^{-x} \geq 0$ for all x. Hence the graph of h is in the 1st and 2nd quadrant.

* $h'(x) = 2x\, e^{-x} + x^2 \cdot \left[e^{-x}(-1)\right] = \underline{e^{-x}\left[2x - x^2\right]} = e^{-x}\, x(2-x)$

sign of h':



∴ $h(x)$ is <u>increasing</u> on $(0,2)$

∴ $h(x)$ is <u>decreasing</u> on $(-\infty, 0)$ and $(2, +\infty)$

∴ local min at $x = 0$; local max at $x = 2$

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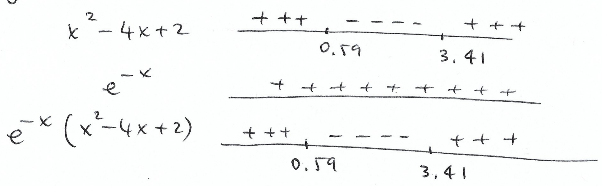
* $h''(x) = (-e^{-x})(2x - x^2) + e^{-x}(2 - 2x) =$

$= e^{-x}\left[-2x + x^2 + 2 - 2x\right] = e^{-x}(x^2 - 4x + 2)$

$h''(x) = 0 \iff e^{-x}(x^2 - 4x + 2) = 0 \iff x^2 - 4x + 2 = 0$

$$x_{1,2} = \frac{4 \pm \sqrt{16 - 8}}{2} = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2} \begin{cases} 2+\sqrt{2} \cong 3.41 \\ 2-\sqrt{2} \cong 0.59 \end{cases}$$

Sign of h'':



∴ h is <u>concave up</u> on $(-\infty, 0.59)$ and $(3.41, +\infty)$

h is concave down on $(0.59, 3.41)$

Notice that $x_1 = 0.59$ and $x_2 = 3.41$ are both inflection points as there is a change of concavity

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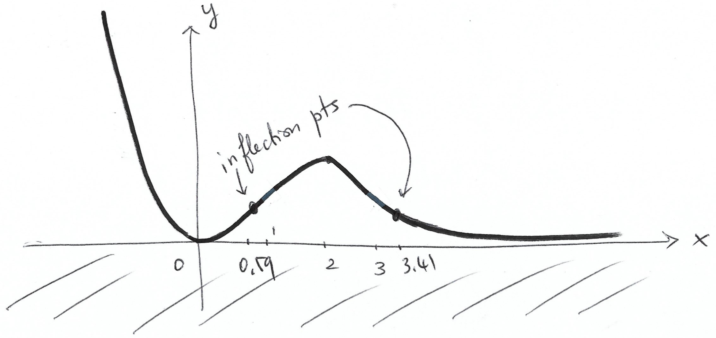
The graph of $h(x)$ looks like:



notice that it is reasonable to expect that $\lim_{x\to\infty} x^2 e^{-x} = 0$. We will see this with L'Hospital rule in Section 5.5

Moreover the function has a <u>global</u> <u>min</u> at $x = 0$; there is just a <u>local max at</u> $x = 2$

Example 4

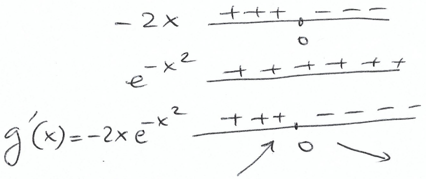
Find the inflection points of the function $g(x) = e^{-x^2}$.

$g(x) = e^{-x^2}$

* Notice that this function is always positive. Hence the graph will be in the first and second quadrant. The graph is also symmetric w.r.t. the y-axis.

* $g'(x) = e^{-x^2} \cdot (-2x) = -2x\, e^{-x^2}$

hence the sign of g' is :



∴ the function g is increasing on $(-\infty, 0)$ and decreasing on $(0, +\infty)$. There is a local max at $x = 0$. (It is actually a global max !)

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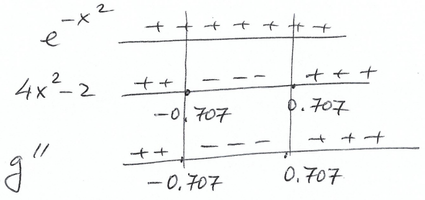
* $g''(x) = -2(1)\, e^{-x^2} - 2x\left[e^{-x^2}(-2x)\right]$

$= e^{-x^2}\left[-2 + 4x^2\right]$

Hence $g''(x) = 0 \iff -2 + 4x^2 = 0 \iff x^2 = \frac{1}{2}$

$\iff x_{1,2} = \pm \frac{\sqrt{2}}{2} = \pm\, 0{,}707$

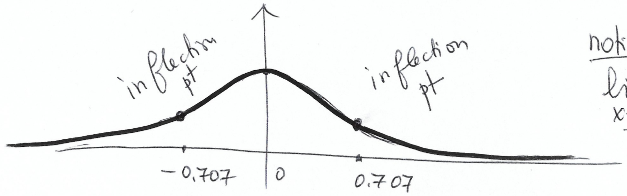
Sign of g'' :



g(x) is conc. up on $(-\infty, -0{,}707)$ and $(0{,}707, +\infty)$

g(x) is conc down on $(-0{,}707, 0{,}707)$

∴ $x_1 = -0{,}707$ and $x_2 = 0{,}707$ are inflection pts



notice : $\lim_{x \to \pm\infty} e^{-x^2} = 0$

Example 5:

Suppose $g(x) = \dfrac{\sqrt{x-3}}{x}$. Find the value of x in the interval $[3, +\infty)$ where $g(x)$ takes its maximum.

The function $g(x) = \dfrac{\sqrt{x-3}}{x}$ is defined on $[3, +\infty)$

Notice that in that interval $g(x) \geq 0$ for all x.

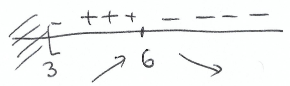
We need to find the intervals of increase and decrease. For that we need to study the sign of $g'(x)$:

$$g'(x) = \frac{\frac{1}{2\sqrt{x-3}} \cdot (1) \cdot x - \sqrt{x-3} \cdot (1)}{x^2} =$$

$$= \frac{\frac{x - 2\left(\sqrt{x-3}\right)^2}{2\sqrt{x-3}}}{x^2} = \frac{x - 2(x-3)}{2x^2\sqrt{x-3}}$$

$$= \frac{x - 2x + 6}{2x^2\sqrt{x-3}} = \frac{6 - x}{2x^2\sqrt{x-3}}$$

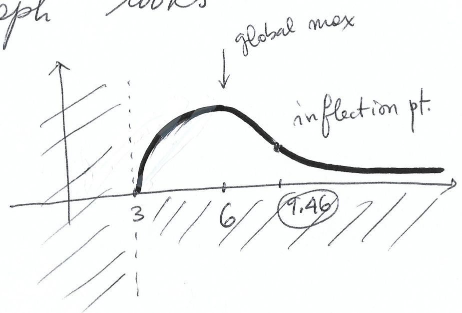
Hence : $g'(x)$



Thus $g(x)$ is increasing on $[3,6)$ and decreasing on $(6,+\infty)$.

Thus $x=6$ is the point where g has a local max. However, because of the behavior of g, this is actually a <u>global</u> max.

The graph looks like:



note $\lim_{x\to\infty} \frac{\sqrt{x-3}}{x} = 0$

From the graph we see that there must be inflection point(s). To find them we need $g''(x)$.

$$g''(x) = \frac{(-1)\left[2x^2\sqrt{x-3}\right] - (6-x)\cdot\left[4x\sqrt{x-3} + 2x^2\frac{1}{2\sqrt{x-3}}\right]}{\left(2x^2\sqrt{x-3}\right)^2}$$

$$= \frac{-4x^2(x-3) - (6-x)8x(x-3) - (6-x)(2x^2)}{4x^4(x-3)\cdot\left[2\sqrt{x-3}\right]}$$

$$= \dots = \frac{3x(x^2-12x+24)}{4x^4(x-3)\sqrt{x-3}}$$

$g''(x)=0$ in $[3,+\infty)$ $\Longleftrightarrow$ $x^2-12x+24=0$ $\Longleftrightarrow$ $x_{1,2} = \frac{12 \pm \sqrt{12^2-4\cdot 24}}{2}$

$= 6 \pm 2\sqrt{3}$

$= \begin{cases} \boxed{9.46} \\ 2.54 \end{cases}$

Example 6: (Exam 3, Fall 13, # 3)

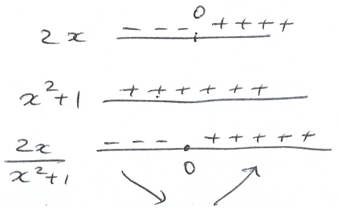
Let $f(x) = \ln(x^2+1)$. You are given that

$$f'(x) = \frac{2x}{x^2+1} \quad \text{and} \quad f''(x) = \frac{2-2x^2}{(x^2+1)^2}.$$

(a) On what intervals is f increasing or decreasing?
(b) At what values of x does f have a local maximum or minimum?
(c) On what intervals is f concave upward or downward?
(d) State the x-coordinate of the inflection point(s) of f.
(e) Use the information in the above to sketch the graph of f.

$f'(x) = \frac{2x}{x^2+1}$

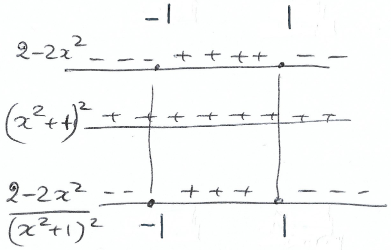
sign of $f'(x)$:



hence f is <u>decreasing</u> on $(-\infty,0)$ and <u>increasing</u> on $(0,+\infty)$

∴ <u>local min</u> at $x=0$

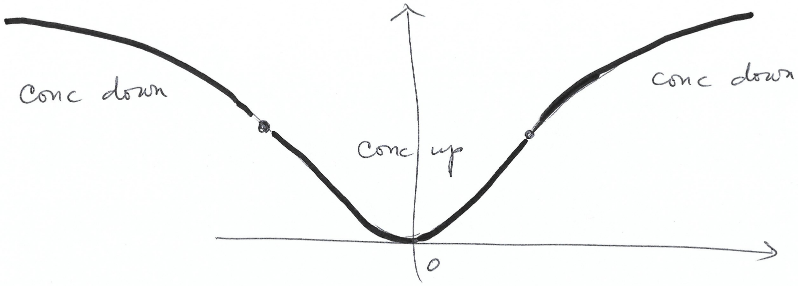
$f'' = \frac{2-2x^2}{(x^2+1)^2}$



∴ f <u>concave down</u> on $(-\infty,-1)$ and $(1,+\infty)$

f is concave up on $(-1, 1)$

There are inflection points at $x_1 = -1$ and $x_2 = 1$



notice that $\lim_{x \to \pm\infty} f(x) = +\infty$

also $x = 0$ is a <u>global</u> minimum

Example 7:

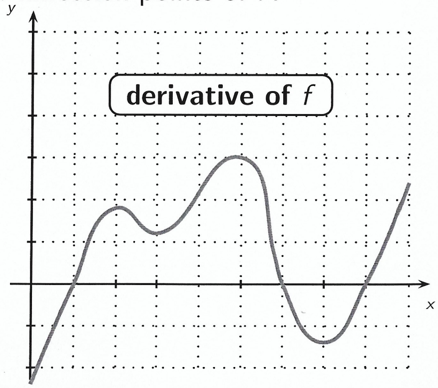
The graph of the <u>derivative</u> f' of a function f is shown.

(a) On what intervals is f increasing or decreasing?

(b) At what values of x does f have a local maximum or minimum?

(c) On what intervals is f concave upward or downward?

(d) State the x-coordinate of the inflection points of f.

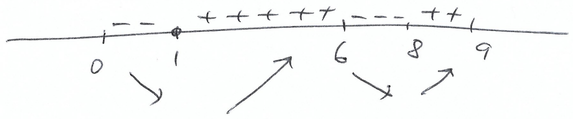


Recall that what we are given is the graph of the <u>derivative</u> of f:

$f' = 0$ from the graph at $x = 1, 6, 8$

the sign of f':

sign f':



f is <u>increasing</u> on $(1, 6)$ and $(8, 9)$

f is <u>decreasing</u> on $(0, 1)$ and $(6, 8)$

<u>local min</u> at $x = 1$; $x = 8$

<u>local max</u> at $x = 6$

To find when f is concave up or concave down we need to know when f' is increasing ($\equiv f$ conc. up) and when f' is decreasing ($\equiv f$ conc. down)

<u>f conc up</u> on $(0, 2)$, $(3, 5)$, $(7, 9)$

<u>f conc down</u> on $(2, 3)$, $(5, 7)$

<u>inflection points</u> at <u>$x = 2, 3, 5, 7$</u>

Example 8: (Online Homework HW18, #14)

Suppose that on the interval I, $f(x)$ is positive and concave up. Furthermore, assume that $f''(x)$ exists and let $g(x) = (f(x))^2$. Use this information to answer the following questions.

(a) $f''(x) >$ ____ on I.
(b) $g''(x) = 2(A^2 + Bf''(x))$, where $A =$ ____ and $B =$ ____
(c) $g''(x) >$ ____ on I.
(d) $g(x)$ is ____ on I.

$f(x)$ is positive and concave up

Hence: $f(x) \geq 0$ and $f''(x) \geq 0$

Consider $g(x) = [f(x)]^2$

$$g'(x) = 2[f(x)]^{2-1} \cdot f'(x) = 2f(x)f'(x)$$

$$\underline{g''(x)} = 2\,\underline{f'(x)} \cdot f'(x) + 2f(x) \cdot f''(x)$$

product rule

$$= 2[f'(x)]^2 + 2f(x) \cdot f''(x)$$

(a) $f''(x) \geq 0$ on I because f is concave up

(b) $A = f'(x)$ and $B = f(x)$

(c) Since $g''(x) = 2[(f'(x))^2 + f(x)f''(x)]$

and $f(x) \geq 0$, $f''(x) \geq 0$ and $(f'(x))^2 \geq 0$

then we have that $\underline{g''(x) \geq 0}$

(d) So g is <u>concave up</u>