MA 137 — Calculus 1 with Life Science Applications
Optimization
(Section 5.4)

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Introduction

There are many situations in which we wish to maximize or minimize certain quantities. For instance,

- in a chemical reaction, you might wish to know under which conditions the reaction rate is maximized;
- in an agricultural setting, you might be interested in finding the amount of fertilizer that would maximize the yield of some crops;
- in a medical setting, you might wish to optimize the dosage of a drug for maximum benefit;
- optimization problems also arise in the study of the evolution of life histories and involve questions such as when an organism should begin reproduction in order to maximize the number of surviving offspring.

In each case, we are interested in finding global extrema.

Suggestions

The most important **skill** in solving a word problem is reading comprehension. The most important **attitude** to have in attacking word problems is to be willing to think about what you are reading and to give up on hoping to *mechanically* apply a set of steps.

MAX-MIN PROBLEMS

All max-min problems ask you to find the largest or smallest value of a function on an interval. Usually, the hard part is reading the English and finding the formula for the function. Once you have found the function, then you can use the techniques from Sections 5.1, 5.2, and 5.3 to find the largest or smallest values.

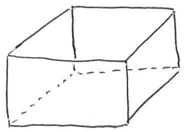
Max-Min Guideline

Nevertheless – despite our previous remarks – we will present some useful strategies to employ that are often helpful.

1. Read the problem
2. Define your variables. If possible, draw a picture and label it.
3. Determine exactly what needs to be maximized or minimized.
4. Write the *general* formula for what you are trying to maximize or minimize. If this formula only involves *one* variable, then skip to step **8**.
5. Find the relationship(s) (i.e., equation(s)) between the variables.
6. Do the algebra to solve for one variable in the equation(s) as a function of the other(s).
7. Use your formula from step **4** to rewrite the formula that you want to maximize or minimize as a function of one variable only.
8. Write down the interval over which the above variable can vary, for the particular word problem you are solving.
9. Take the derivative and find the critical points.
10. Use the techniques from Chapter 5 to find the maximum or the minimum.

Example 1: (Online Homework HW19, # 1)

Find the dimensions of an open rectangular box with a square base that holds 7000 cubic cm and is constructed with the least building material possible.



Let x be the size of one of the sides of the square base. Let y be the height of the box.

We know that $\boxed{x^2 \cdot y \overset{\text{MUST}}{=} 7000}$

Hence $y = \frac{7000}{x^2}$

The box has no top and we need to minimize the surface area:

$$\text{Surface area} = S = \underbrace{4xy}_{\text{4 sides}} + \underbrace{x^2}_{\text{base}} = 4x\frac{7000}{x^2} + x^2$$

$$\therefore \boxed{S(x) = \frac{28,000}{x} + x^2}$$

There are no constraints on x other than

$$0 < x < +\infty$$

it is an open interval.

Since $x>0$ then $S(x)$ only takes positive values

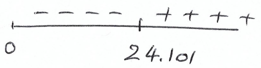
That is $S(x)>0$.

$$S'(x) = -\frac{28,000}{x^2} + 2x = \frac{2x^3-28,000}{x^2}$$

$$S'(x)=0 \iff 2x^3-28,000=0 \iff x^3=14,000$$

$$\iff x = \sqrt[3]{14000} = \underline{\underline{24.101}}$$

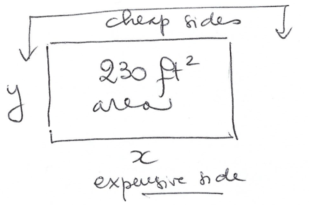
sign of $S'(x)$



Hence $S(x)$ is decreasing before 24.101 and increasing after 24.101. Hence the value $\boxed{x=24.101}$ is a point of a local min. Since there is just this value, it is actually a global min for $S(x)$

Example 2: (Online Homework HW19, # 4)

A fence is to be built to enclose a rectangular area of 230 square feet. The fence along three sides is to be made of material that costs 3 dollars per foot, and the material for the fourth side costs 13 dollars per foot. Find the dimensions of the enclosure that is most economical to construct.



Let x and y be the sizes of the sides of the rectangular area : $xy = 230$

So $y = \frac{230}{x}$ $\quad$ $0 < x < \infty$

$$\text{Cost of enclosure} = \underbrace{3\cdot(2y+x)}_{\text{cheaper side}} + \underbrace{13\cdot x}_{\text{expensive side}}$$

$$C(x) = 6\left(\frac{230}{x}\right) + 16x = \frac{1380}{x} + 16x \qquad 0<x<\infty$$

We need to minimize $C(x)$.

$$C'(x) = -\frac{1380}{x^2} + 16 = \frac{16x^2-1380}{x^2}$$

only positive

$$C'(x)=0 \iff 16x^2-1380=0 \iff x = \pm\sqrt{\frac{1380}{16}} \cong \pm 9.287$$ feet

sign of C' $\quad \underset{9.287}{- - - \,|\, + + +}$

Thus $C(x)$ has a local minimum at $x = 9.287$

$\underset{\searrow\ 9.287\ \nearrow}{- - - - \,|\, + + +}$

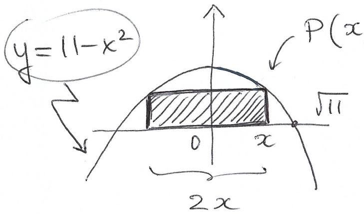
Since this is the only extremum, then $C(x)$ has a global minimum.

the other dimension of the enclosure is

$$y = \frac{230}{9.287} = \underline{24.766\ ft}$$

Example 3: (Online Homework HW19, # 5)

A rectangle is inscribed with its base on the x-axis and its upper corners on the parabola $y = 11 - x^2$. What are the dimensions of such a rectangle with the greatest possible area?



If x denotes a point between 0 and $\sqrt{11}$ then the height of the rectangle at $P(x, 11-x^2)$ is exactly $11-x^2$

Thus the area of the whole rectangle is

$$A(x) = \underbrace{2x}_{\text{base}}\cdot\underbrace{(11-x^2)}_{\text{height}} \qquad 0\le x\le\sqrt{11}$$

Hence we need to maximize $A(x) = 22x - 2x^3$ on the closed interval $[0, \sqrt{11}]$. Since $A(x)$ is continuous on the interval, we know that a global max exists by the Extreme Value Theorem

We need to check the value of $A(x)$ at

- end points of $[0, \sqrt{11}]$
- critical points of $A(x)$

Obviously, at the end points of $[0, \sqrt{11}]$ the area of the rectangle is 0.

$$A'(x) = 22 - 6x^2 = 0 \iff x^2 = \frac{22}{6}$$

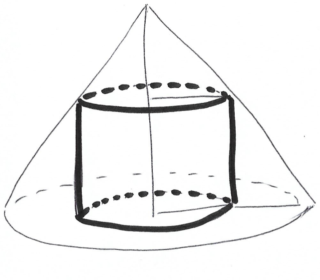
$\iff \underline{x = \pm 1.915}$ But x is positive for our choice so $\boxed{\underline{x_0 = 1.915}}$

$A(1.915) \cong 28.0845 \longleftarrow$ global max at this value

The dimensions are: $\underline{\underline{2x_0}}$ times $\underline{\underline{y_0}}$: 3.83 times 7.333

Example 4: (Online Homework HW19, # 6)

A cylinder is inscribed in a right circular cone of height 6 and radius (at the base) equal to 2.5. What are the dimensions of such a cylinder which has maximum volume?

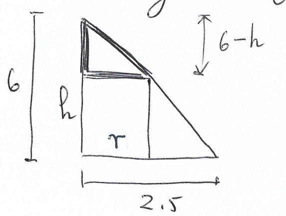


The height of the cone is 6 and the radius of the cone is 2.5

Let r be the radius of the cylinder and h its height.

$0 \le r \le 2.5$

We need to maximize the volume of the cylinder $V = \pi r^2 \cdot h$. We need to eliminate h from the expression. We use the similar triangles from the picture:



$$\frac{6}{2.5} = \frac{6-h}{r} \quad \text{OR}$$

$$2.4r = 6 - h \quad \text{OR}$$

$$\boxed{h = 6 - 2.4r}$$

$\underline{\text{Thus}}$: $V(r) = \pi r^2 (6 - 2.4r) \qquad 0 \le r \le 2.5$

$$= 6\pi r^2 - 2.4\pi r^3$$

Since $V(r)$ is continuous on the closed interval $[0, 2.5]$, by the EVT the function $V(r)$ has a global maximum. The candidates where the global max is attained are: end points or critical points.

- $V(0) = 0 = V(2.5)$
- $V'(r) = 0 \iff 12\pi r - 7.2\pi r^2 = 0$

$\iff r = 0$ OR $12\pi - 7.2\pi r = 0$

$\underline{V(1.667) = 17.453 \text{ unit}^3}$ global max

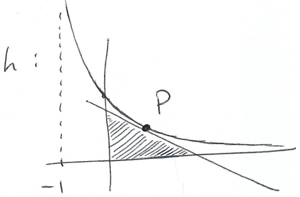
$\underline{\underline{r = 1.667}}$

$\underline{\underline{h = 1.9992}}$

Example 5: (Online Homework HW19, # 8)

Find the maximum area of a triangle formed in the first quadrant by the x-axis, y-axis and a tangent line to the graph of $f(x) = (x+1)^{-2}$.

$f(x) = \frac{1}{(x+1)^2}$ has the following graph:



Let P a point on graph of f.

Let $x=a$ be its x-coordinate; that $y = \frac{1}{(a+1)^2}$.

We need to find the equation of the tg line at P.

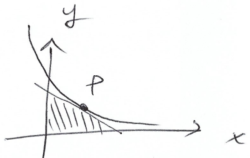
$f'(x) = -2(x+1)^{-3} = \frac{-2}{(x+1)^3}$ so $f'(a) = \frac{-2}{(a+1)^3}$

$$\therefore \quad \boxed{y - \frac{1}{(a+1)^2} = \frac{-2}{(a+1)^3}(x-a)}$$

OR $y = -\frac{2}{(a+1)^3}x + \frac{2a}{(a+1)^3} + \frac{1}{(a+1)^2}$

$= -\frac{2}{(a+1)^3}x + \frac{3a+1}{(a+1)^3}$

The y-intercept of the tangent line is $y = \frac{3a+1}{(a+1)^3}$; the x-intercept of the tangent line is $\frac{3a+1}{2}$



Thus the area of the triangle is

$$A(a) = \frac{3a+1}{2} \cdot \frac{(3a+1)}{(a+1)^3} \cdot \frac{1}{2} = \frac{(3a+1)^2}{4(a+1)^3}$$

the x-coordinate of the point P varies between -1 and ∞.

Thus we need to maximize

$$\boxed{A(a) = \frac{(3a+1)^2}{4(a+1)^3} \quad \text{on} \quad -1 < a < \infty}$$

For simplicity, let's change the name of the variable:

$$A(x) = \frac{(3x+1)^2}{4(x+1)^3} \qquad -1 < x < \infty$$

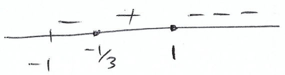
We need do study the sign of $A'(x)$;

$$A'(x) = \frac{2(3x+1)\cdot(3)[4(x+1)^3] - (3x+1)^2\cdot[4\cdot 3(x+1)^2\cdot(1)]}{16(x+1)^6}$$

$$= \frac{12(3x+1)\cancel{(x+1)^2}[2(x+1) - (3x+1)]}{16(x+1)^{\cancel{6}\,4}} = \underline{\underline{\frac{12(3x+1)(1-x)}{16(x+1)^4}}}$$

$A'(x) = 0 \iff \underline{\underline{x = -1/3}}$ OR $\underline{\underline{x = 1}}$

sign of A':



thus there is a maximum at $\underline{x=1}$

$$\boxed{A(1) = 1/2}$$

Example 6: (≈ Neuhauser, Example 1, p. 243)

Let $Y(N)$ be the yield of an agricultural crop as a function of nitrogen level N in the soil. A model that is used for this relationship is

$$Y(N) = \frac{N}{1+N^2} \qquad \text{for } N \geq 0$$

(where N is measured in appropriate units). Find the nitrogen level that maximizes yield.

$Y(N) = \frac{N}{1+N^2}$, with $N \geq 0$

[If you do not like the name of the variable switch to $y(x) = \frac{x}{1+x^2}$.]

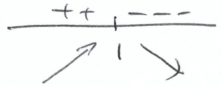
Notice that $Y(N) \geq 0$ for $N \geq 0$, as it should be since $Y(N)$ denotes a yield.

Let's find $Y'(N)$. $Y'(N) = \frac{1\cdot(1+N^2) - N(2N)}{(1+N^2)^2} =$

$= \frac{1+N^2-2N^2}{(1+N^2)^2} = \frac{1-N^2}{(1+N^2)^2}$

Notice that $Y'(N) = 0 \iff 1-N^2 = 0 \iff N = \pm 1$. Hence $\boxed{N=1}$ since $Y(N)$ is defined for $N \geq 0$.

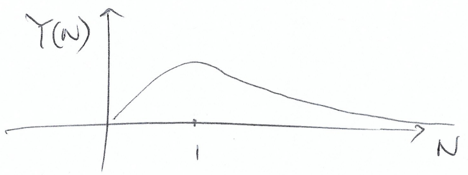
Also sign Y':



Hence $N=1$ is a point where $Y(N)$ has a local maximum. It is actually a global max, since there is only one critical value.

Notice $\lim_{N\to\infty} Y(N) = \lim_{N\to\infty} \frac{N}{1+N^2} = 0$

graph of $Y(N)$ looks like



Example 7:

Suppose that a patient is given a dosage x of some medication, and the probability of a cure is

$$P(x) = \frac{\sqrt{x}}{1+x}.$$

What dosage maximizes the probability of a cure?

$$P(x) = \frac{\sqrt{x}}{1+x} \qquad 0 < x < +\infty$$

We need to find the critical numbers and study the sign of $P'(x)$:

$$P'(x) = \frac{\frac{1}{2\sqrt{x}}(1+x) - \sqrt{x}(1)}{(1+x)^2} = \frac{\frac{1+x-(2\sqrt{x})(\sqrt{x})}{2\sqrt{x}}}{(1+x)^2}$$

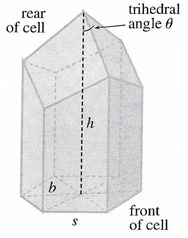
$$= \frac{1+x-2x}{2\sqrt{x}(1+x)^2} = \frac{1-x}{2\sqrt{x}(1+x)^2}$$

$$P'(x) = 0 \iff 1-x=0 \quad \text{so} \quad \boxed{x=1}$$

sign $P'(x)$: $\underset{P(x)\nearrow}{+++}\ \underset{1}{|}\ \underset{\searrow P(x)}{----}$

Thus $x=1$ is a local maximum. Actually it is a global maximum since there is only one critical value.

Example 8: (Online Homework HW19, # 11)



In a beehive, each cell is a regular hexagonal prism, open at one end with a trihedal angle at the other end. It is believed that bees form their cells in such a way as to minimize the surface area for a given volume, thus using the least amount of wax in cell construction. Examination of these cells has shown that the measure of the apex angle θ is amazingly consistent.

Based on the geometry of the cell, it can be shown that the surface area S is given by

$$S = 6sh - \frac{3}{2}s^2\cot\theta + \frac{3\sqrt{3}}{2}s^2\csc\theta$$

where s, the length of the sides of the hexagon, and h, the height, are constants.

(a) Calculate $dS/d\theta$.

(b) What angle should bees prefer (in radians)?

(c) Determine the minimum surface area of the cell.

$$S = 6sh - \frac{3}{2}s^2\underline{\underline{\cot\theta}} + \frac{3\sqrt{3}}{2}s^2\underline{\underline{\csc\theta}}$$

s, h are constant

$$\frac{dS}{d\theta} = -\frac{3}{2}s^2\frac{d}{d\theta}\left(\frac{\cos\theta}{\sin\theta}\right) + \frac{3\sqrt{3}}{2}s^2\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\right)$$

$$= -\frac{3}{2}s^2\frac{(-\sin\theta)\sin\theta - \cos\theta(\cos\theta)}{\sin^2\theta} + \frac{3\sqrt{3}}{2}s^2\frac{(-\cos\theta)}{\sin^2\theta}$$

$$= -\frac{3}{2}s^2\frac{(-1)}{\sin^2\theta} - \frac{3\sqrt{3}s^2\cos\theta}{2\sin^2\theta}$$

$$= \boxed{\frac{3s^2(1-\sqrt{3}\cos\theta)}{2\sin^2\theta}}$$

$$\frac{dS}{d\theta} = 0 \iff 1-\sqrt{3}\cos\theta = 0 \iff \cos\theta = \frac{1}{\sqrt{3}}$$

Using your calculator $\cos\theta = \frac{1}{\sqrt{3}} \iff$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = 0.9553$$

sign of $\frac{dS}{d\theta}$: $\underset{\searrow}{----}\cdot\underset{\nearrow}{+++}$

Hence this is the value that gives a local min.

Again this is a global min.

When $\cos\theta = \frac{1}{\sqrt{3}}$ $\quad \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{1}{3}} = \sqrt{\frac{2}{3}}$

Hence

$$S = \text{minimal surface area} = 6sh - \frac{3}{2}s^2\frac{1/\sqrt{3}}{\sqrt{2}/\sqrt{3}} + \frac{3\sqrt{3}}{2}s^2\cdot\frac{1}{\sqrt{2/3}}$$

∴ the minimal surface area is

$$S = 6sh - \frac{3}{2}s^2\frac{1}{\sqrt{2}} + \frac{3\sqrt{3}\,s^2}{2}\cdot\frac{\sqrt{3}}{\sqrt{2}}$$

$$= 6sh - \frac{3}{2}s^2\frac{1}{\sqrt{2}} + \frac{9s^2}{2\sqrt{2}}$$

$$= \boxed{6sh + \frac{3s^2}{\sqrt{2}}}$$

Example 9: (Online Homework HW19, # 12)

One question for fishery management is how to control fishing to optimize profits for the fishermen. One DE describing the population dynamics for a population of fish F with harvesting is given by the equation,

$$\frac{dF}{dt} = rF\left(1 - \frac{F}{K}\right) - xF$$

where r is the growth rate of this species of fish at low density, K is the carrying capacity of this population, and x is the harvesting effort of the fishermen. The non-zero equilibrium of this equation is given by

$$F_e = K\frac{(r-x)}{r}.$$

One formula for profitability is computed by the equation

$$P(x) = xF_e = Kx\frac{(r-x)}{r}.$$

Find the level of harvesting x that produces the maximum profit possible $x_{\max}$ with this dynamics.

What is the equilibrium population F_e at this optimal profitability?

Also, determine the maximum possible fish population for this model and at what harvesting level this occurs.

$$P(x) = Kx\left(\frac{r-x}{r}\right) = Kx - \frac{K}{r}x^2$$

$$P'(x) = K - \frac{K}{r}\cdot 2x = 0$$

$$\iff K = \frac{K}{r}2x \iff \boxed{x = \frac{r}{2}}$$

sign of $P'(x)$: $\underset{r/2}{+++\,|\,---}$

∴ there is a local max when $x = r/2$

Since this is the only critical value it gives a global max at $x = r/2$

F_e when $x = r/2$ is $K\cdot\frac{(r-(r/2))}{r} = K\frac{r/2}{r} = \boxed{\frac{K}{2}}$

Of course the maximum population in this model would occur when there is no harvesting: $\underline{x=0}$. In this case

$\frac{dF}{dt} = rF(1-F/K)$ is the logistic growth model. The max occurs when we reach the carrying capacity K.

Example 10: (Online Homework HW19, # 14)

[From: D. A. Roff, The Evolution of Life Histories, Chapman and Hall, 1992.]

Semelparous organisms breed only once during their lifetime. Examples of this type of reproduction strategy can be found with Pacific salmon and bamboo. The per capita rate of increase, r, can be thought of as a measure of reproductive fitness. The greater r, the more offspring an individual produces. The intrinsic rate of increase is typically a function of age, x. Models for age-structured populations of semelparous organisms predict that the intrinsic rate of increase as a function of x is given by

$$r(x) = \frac{\ln[L(x)M(x)]}{x},$$

where $L(x)$ is the probability of surviving to age x and $M(x)$ is the number of female births at age x. Suppose that

$$L(x) = e^{-0.17x} \quad \text{and} \quad M(x) = 3x^{0.7}.$$

Find the optimal age of reproduction.

$$r(x) = \frac{\ln(L(x)M(x))}{x} = \frac{\ln(e^{-0.17x} \cdot 3x^{0.7})}{x} = \frac{\ln e^{-0.17x} + \ln(3x^{0.7})}{x} = \frac{-0.17x + (\ln 3) + \ln(x^{0.7})}{x} = \boxed{-0.17 + \frac{\ln(3) + 0.7\ln x}{x}} \quad 0 < x < \infty$$

We need to find $r'(x) = 0$ and study the sign of $r'(x)$.

$$r(x) = -0.17 + \frac{\ln(3) + 0.7\ln x}{x}$$

$$r'(x) = 0 + \frac{(0.7\frac{1}{x})x - (\ln(3) + 0.7\ln x)\cdot(1)}{x^2}$$

$$= \frac{0.7 - \ln(3) - 0.7\ln x}{x^2}$$

$$r'(x) = 0 \iff 0.7 - \ln(3) - 0.7\ln x = 0$$

$$\iff \ln x = \frac{0.7 - \ln(3)}{0.7} \cong -0.5694$$

$$\iff x = e^{-0.5694} \cong \boxed{0.5658}$$ optimal reproduction age

sign $r'(x)$: $++$ | $---$ ↗ 0.5658 ↘