MA 137 — Calculus 1 with Life Science Applications
L'Hôpital's Rule
(Section 5.5)

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Heuristics

We have often encountered the situation in which we had to compute $\lim_{x\to a}\frac{f(x)}{g(x)}$ and we had that both the following limits were zero

$$\lim_{x\to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to a} g(x) = 0.$$

Using a linear approximation at $x = a$, we find that, for x close to a

$$\frac{f(x)}{g(x)} \approx \frac{f(a)+f'(a)(x-a)}{g(a)+g'(a)(x-a)}$$

Since $f(a) = g(a) = 0$ and $x \neq a$, the right-hand side is equal to

$$\frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}$$

provided that $f'(a)/g'(a)$ is defined. We therefore hope that something like

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

holds when $f(a)/g(a)$ is of the form $0/0$ and $f'(a)/g'(a)$ is defined. In fact, something like this does hold; it is called **l'Hôpital's rule**.

L'Hôpital's Rule

Theorem

Suppose that f and g are differentiable functions and that

$$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x) \qquad \textit{or} \qquad \lim_{x\to a} f(x) = \infty = \lim_{x\to a} g(x)$$

Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided the second limit exists.

L'Hôpital's rule can actually be applied to calculate limits for seven kinds of indeterminate expressions

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0\cdot\infty \qquad \infty-\infty \qquad 0^0 \qquad 1^\infty \qquad \infty^0.$$

(Note that l'Hôpital's rule works for $a = +\infty$ or $-\infty$ as well.)

Reduction to $0/0$ or ∞/∞ Form

$0\cdot\infty$ Suppose we have to compute $\lim_{x\to a} f(x)g(x)$ where $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$. To apply l'Hôpital's rule to this kind of limit write it in one of the two forms

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a}\frac{f(x)}{1/g(x)} = \lim_{x\to a}\frac{g(x)}{1/f(x)}$$

In the first case the ratio is $0/0$, whereas in the second case the ratio is ∞/∞. Usually only one of the two expressions is easy to evaluate.

$\infty-\infty$ Suppose we have to compute $\lim_{x\to a}[f(x)-g(x)]$ where $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$. To apply l'Hôpital's rule to this kind of limit write it in one of the two forms

$$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x)\left(1-\frac{g(x)}{f(x)}\right) = \lim_{x\to a} g(x)\left(\frac{f(x)}{g(x)}-1\right)$$

and hope that the limit is of the form $0\cdot\infty$.

$0^0\ 1^\infty\ \infty^0$ Suppose we have to compute $\lim_{x\to a}[f(x)]^{g(x)}$, which becomes of the form 0^0, 1^∞ or ∞^0. The key to solving these limits is to write them as exponentials

$$\lim_{x\to a}[f(x)]^{g(x)} = \lim_{x\to a}\exp\left\{\ln[f(x)]^{g(x)}\right\} = \lim_{x\to a}\exp\left\{g(x)\cdot\ln f(x)\right\} = \exp\left[\lim_{x\to a}(g(x)\cdot\ln f(x))\right].$$

The last step, in which we interchanged lim and exp, uses the fact that the exponential function is continuous.

Example 1: (Nuehauser, p. 253)

Evaluate $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3}$.

$\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3} = \dfrac{0}{0}$ if we use direct evaluation

Hence we can apply l'Hôpital's rule.

We obtain:

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{2x}{1} = \frac{6}{1} = \boxed{6}$$

Note: in Chapter 3 we solved the problem by factoring and simplifying the expression

$$\lim_{x \to 3} \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}} = \lim_{x \to 3} x+3 = 3+3 = 6$$

Example 2: (Nuehauser, p. 253)

Evaluate $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x}$.

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \frac{e^0 - 1}{0} = \frac{0}{0}$$

Hence we can use l'Hôpital's rule:

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x}{1} = e^0 = \boxed{1}$$

Example 3: (Neuhauser, Example 3, p. 255)

Evaluate $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x^2}$.

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1-1}{0} = \frac{0}{0} \quad \text{by direct substitution}$$

Hence we can use l'Hôpital's rule

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{-(-\sin x)}{2x} = \lim_{x\to 0}\frac{\sin x}{2x}$$

$= \frac{0}{0}$ again. Hence we use l'Hôpital's rule again

$$= \lim_{x\to 0}\frac{\cos x}{2} = \boxed{\frac{1}{2}}$$

Note that in Section 3.4 we gave a geometric argument for $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$

Example 4: (Neuhauser, Problem # 25, p. 259)

Evaluate $\displaystyle\lim_{x\to\infty} x\cdot e^{-x}$.

What about $\displaystyle\lim_{x\to\infty} x^{13}\cdot e^{-x}$? (Online Homework HW20, # 5)

$$\lim_{x\to\infty} x e^{-x} = \lim_{x\to\infty}\frac{x}{e^x} = \frac{\infty}{\infty}$$

We can use l'Hôpital rule. Hence

$$\lim_{x\to\infty}\frac{x}{e^x} = \lim_{x\to\infty}\frac{1}{e^x} = \frac{1}{\infty} = \boxed{0}$$

HW20, #5: $\displaystyle\lim_{x\to\infty}\frac{x^{13}}{e^x} = \frac{\infty}{\infty}$ = use l'Hôpital's rule $= \displaystyle\lim_{x\to\infty}\frac{13x^{12}}{e^x} = \frac{\infty}{\infty} = \dots =$ use many more times l'Hôpital's rule to get

$$= \lim_{x\to\infty}\frac{13!}{e^x} = \frac{13!}{\infty} = \boxed{0}$$

Example 5: (Online Homework HW20, # 3)

Evaluate $\lim_{x\to 0^+} 7\sqrt{x}\cdot \ln x$.

$\lim_{x\to 0^+} 7\sqrt{x}\cdot \ln x = 0\cdot(-\infty)$

Hence we can use l'Hôpital's rule, where we rewrite $7\sqrt{x}\ln x$ as $\dfrac{7\ln x}{1/\sqrt{x}} = \dfrac{7\ln x}{x^{-1/2}}$

Hence:

$$\lim_{x\to 0^+} 7\sqrt{x}\cdot\ln x = \lim_{x\to 0^+}\frac{7\ln x}{x^{-1/2}} = \frac{-\infty}{+\infty} =$$

$$= \text{l'Hôpital} = \lim_{x\to 0^+}\frac{7\,\frac{1}{x}}{-\frac{1}{2}x^{-3/2}} = \lim_{x\to 0^+} 7\frac{1}{x}\left[-2x^{3/2}\right]$$

$$= \lim_{x\to 0^+} -14\,\frac{x\sqrt{x}}{x} = \lim_{x\to 0^+} -14\sqrt{x} = 0$$

Hence $\boxed{\lim_{x\to 0^+} 7\sqrt{x}\,\ln x = 0}$

Example 6: (Neuhauser, Example 9, p. 257)

Evaluate $\lim_{x\to\infty} x - \sqrt{x^2+x}$.

$\lim_{x\to\infty} x - \sqrt{x^2+x} = \infty - \infty$

Let's rewrite the limit as follows:

$$\lim_{x\to\infty} x - \sqrt{x^2+x} = \lim_{x\to\infty} x\left[1 - \frac{\sqrt{x^2+x}}{x}\right] =$$

$$= \lim_{x\to\infty}\frac{1-\sqrt{1+\frac{1}{x}}}{\frac{1}{x}} = \frac{0}{0} = \text{hence we can apply l'Hôpital's rule}$$

$$= \lim_{x\to\infty}\frac{-\frac{1}{2}\left(1+\frac{1}{x}\right)^{-1/2}\cdot\cancel{\left(-1/x^2\right)}}{\cancel{-1/x^2}} = \lim_{x\to\infty}\frac{-1/2}{\sqrt{1+1/x}} = \boxed{-\frac{1}{2}}$$

Example 7: (Online Homework HW20, # 4)

Evaluate $\lim_{x\to 0^+} x^x$.

$$\lim_{x\to 0^+} x^x = 0^0$$

Hence we can rewrite the limit as:

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln x^x} = \lim_{x\to 0^+} e^{x\ln x}$$

$$= e^{\lim_{x\to 0^+} x \ln x} = e^0 = \boxed{1}$$

<u>But</u> $\lim_{x\to 0^+} x\ln x = 0\cdot(-\infty) = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \frac{-\infty}{\infty}$

$$= \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} \frac{1}{x}\cdot(-x^2) = \lim_{x\to 0^+}(-x) = \underline{\underline{0}}$$

Example 8: (Neuhauser, Problem 62, p. 259)

Use l'Hôpital's rule to find $\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x$ where c is a constant.

What about $\lim_{x\to\infty} 3x\big(\ln(x+3)-\ln x\big)$? (Online Homework HW20, #10)

$$\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x = 1^\infty$$

Hence we can rewrite the limit as

$$\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{c}{x}\right)^x} =$$

$$= \lim_{x\to\infty} e^{x\ln\left(1+\frac{c}{x}\right)} = e^{\lim_{x\to\infty}\frac{\ln\left(1+\frac{c}{x}\right)}{\frac{1}{x}}} = \boxed{e^c}$$

<u>Note</u> that $\lim_{x\to\infty}\frac{\ln\left(1+\frac{c}{x}\right)}{\frac{1}{x}} = \frac{0}{0} =$ by l'Hôpital's rule

$$= \lim_{x\to\infty}\frac{\frac{1}{1+c/x}\cdot\left(-\frac{c}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{c}{1+c/x} = \lim_{x\to\infty}\frac{cx}{x+c} = \boxed{c}$$

$$\lim_{x\to\infty} 3x\left[\ln(x+3) - \ln x\right] = \infty(\infty-\infty)$$

$$= \lim_{x\to\infty} 3x \cdot \ln\left(\frac{x+3}{x}\right) = \lim_{x\to\infty} 3 \cdot \ln\left(\frac{x+3}{x}\right)^x =$$

$$= 3 \cdot \ln\left[\underbrace{\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x}_{e^3}\right] = 3 \ln e^3 = 3 \cdot 3 = 9$$

(by the first part)