MA 137 – Calculus 1 with Life Science Applications
A First Look at Differential Equations
(Section 5.9)

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Differential Equations ($\equiv$ DEs)

A **differential equation** is an equation that contains an unknown function and one (or more) of its derivatives.

For example

- $\dfrac{dy}{dx} + 6y = 7$;
- $\dfrac{dy}{dt} + 0.2\,t\,y = 6t$;
- $\dfrac{dP}{dt} = \sqrt{P\,t}$;
- $xy' + y = y^2$.

If a differential equation contains only the first derivative, it is called a **first-order differential equation**: $\dfrac{dy}{dx} = h(x,y)$.

Example 1

Consider the differential equation $(t+1)\dfrac{dy}{dt} - y + 6 = 0$.

Which of the following functions

$$y_1(t) = t+7 \qquad y_2(t) = 3t+21 \qquad y_3(t) = 3t+9$$

are solutions for all t?

$$(t+1)\frac{dy}{dt} - y + 6 = 0$$

(1.) Consider $y_1 = t+7$; $\dfrac{dy_1}{dt} = 1$ and now plug into the equation

$$(t+1)\cdot(1) - (t+7) + 6 = t + 1 - t - 7 + 6 = 0 \checkmark$$

(2.) Consider $y_2 = 3t+21$; $\dfrac{dy_2}{dt} = 3$ and now plug into the equation

$$(t+1)\cdot(3) - [3t+21] + 6 =$$

$$= \cancel{3t} + 3 - \cancel{3t} - 21 + 6 = -12 \neq 0$$

(3.) Consider $y_3 = 3t+9$; $\dfrac{dy_3}{dt} = 3$ and now plug into the equation

$$(t+1)(3) - (3t+9) + 6 =$$

$$= \cancel{3t} + \cancel{3} - \cancel{3t} - \cancel{9} + \cancel{6} = 0 \checkmark$$

Hence $\boxed{y_1 \text{ and } y_3}$ are both solutions of the differential equation

$$\boxed{(t+1)\frac{dy}{dt} - y + 6 = 0}$$

Example 2

Verify that the function

$$y = 1 - \frac{1}{x^3 + C} \qquad C = \text{any constant}$$

is a (family of) solution(s) of the differential equation

$$y' - 3(y-1)^2 x^2 = 0.$$

Verify that $y = 1 - \dfrac{1}{x^3+C}$ is a family of solutions of the DE:

$$y' - 3(y-1)^2 x^2 = 0$$

We need y'. Using the quotient rule

$$y' = 0 - \frac{0 \cdot (x^3+C) - 1 \cdot (3x^2)}{(x^3+C)^2} = \frac{+3x^2}{(x^3+C)^2}$$

Hence, let us plug into the equation

$$\left[\frac{3x^2}{(x^3+C)^2}\right] - 3\left[\cancel{1} - \frac{1}{x^3+C} - \cancel{1}\right]^2 x^2 =$$

$$\frac{3x^2}{(x^3+C)^2} - \frac{3x^2}{(x^3+C)^2} = 0 \checkmark \quad \text{Yes!}$$

DEs arise for example in biology (e.g. models of population growth), economics (e.g. models of economic growth), and many other areas.

exponential growth model:	$\dfrac{dN}{dt} = rN \quad N(0) = N_0;$
logistic growth model:	$\dfrac{dN}{dt} = rN\left(1 - \dfrac{N}{K}\right) \quad N(0) = N_0;$
Newton's law of cooling:	$\dfrac{dT}{dt} = -k(T - T_e) \quad T(0) = T_0;$
von Bertalanffy models:	$\dfrac{dL}{dt} = k(L_\infty - L) \quad L(0) = L_0;$
	$\dfrac{dW}{dt} = \eta W^{2/3} - \kappa W \quad W(0) = W_0;$
Solow's economic growth model:	$\dfrac{dk}{dt} = sk^\alpha - \delta k \quad k(0) = k_0.$

The Exponential Growth Model

A biological population with plenty of food, space to grow, and no threat from predators, tends to grow at a rate that is proportional to the population – that is, in each unit of time, a certain percentage of the individuals produce new individuals.

If reproduction takes place more or less continuously, then this growth rate is represented by

$$\frac{dN}{dt} = rN,$$

where $N = N(t)$ is the population as a function of time t and r is the growth rate.

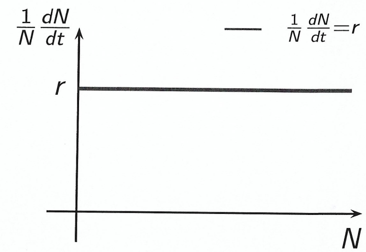
Assume also that N_0 is the population at time $t = 0$.

Note: $r =$ birth rate $-$ mortality rate.

Rewriting this differential equation as

$$\frac{1}{N}\frac{dN}{dt} = r$$

says that the per capita growth rate in the exponential model is a constant function of population size.



We will show (later) that the solution to this differential equation is

$$\boxed{N(t) = N_0 e^{rt}}.$$

The Logistic Growth Model ($\equiv$ Verhulst Model)

- In short, unconstrained natural growth is exponential growth.
- However, we may account for the growth rate declining to 0 by including a factor $1 - N/K$ in the model, where K is a positive constant.
- The factor $1 - N/K$ is close to 1 (that is, has no effect) when N is much smaller than K, and is close to 0 when N is close to K.
- The resulting model,

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \qquad \text{with} \qquad N(0) = N_0$$

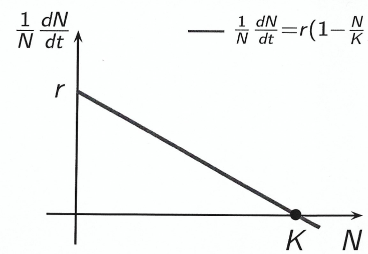
is called the **logistic growth model** or the **Verhulst model**.

The word "logistic" has no particular meaning in this context, except that it is commonly accepted. The second name honors **Pierre François Verhulst** (1804–1849), a Belgian mathematician who studied this idea in the 19th century. Using data from the first five U.S. censuses, he made a prediction in 1840 of the U.S. population in 1940 – and was off by less than 1%.

Rewriting this differential equation as

$$\frac{1}{N}\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)$$

says that the per capita growth rate in the logistic equation is a linearly decreasing function of population size.



Note: r ($\equiv$growth rate) and K ($\equiv$carrying capacity) are positive constants.

We will show (later) that the solution to this differential equation is

$$\boxed{N(t) = \frac{K}{1 + \left(\frac{K}{N_0} - 1\right)e^{-rt}}}$$

Observe that $\lim_{t\to\infty} N(t) = K$.

This justifies that the constant K is dubbed **carrying capacity**.

Compare the logistic growth DE

$$\boxed{\frac{1}{N}\cdot\frac{dN}{dt} = r\left(1-\frac{N}{K}\right)}$$

to the discrete logistic model:

$$N_{t+1} = N_t\left[1 + R\left(1-\frac{N_t}{K}\right)\right] \quad \text{for } t=0,1,2,\ldots$$

We can rewrite the latter as:

$$N_{t+1} = N_t + R N_t\left(1-\frac{N_t}{K}\right)$$

$$\Longleftrightarrow$$

$$\frac{N_{t+1}-N_t}{1} = R N_t\left(1-\frac{N_t}{K}\right)$$

$$\Longleftrightarrow$$

$$\boxed{\frac{1}{N_t}\left(\frac{N_{t+1}-N_t}{1}\right) = R\left(1-\frac{N_t}{K}\right)}$$

!!

Newton's Law of Cooling

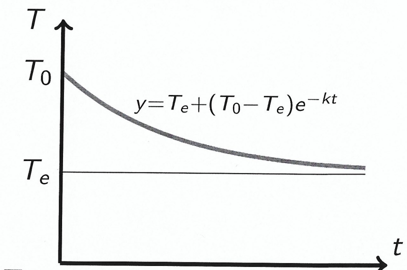
It states that the rate at which an object cools is proportional to the difference in temperature between the object and the surrounding medium:

$$\frac{dT}{dt} = -k(T - T_e) \qquad T(0) = T_0,$$

where k is a positive constant.

We can show that the solution of this IVP is given by

$$\boxed{T(t) = T_e + (T_0 - T_e)e^{-kt}}.$$



Notice also that

$$\lim_{t\to\infty} T(t) = \lim_{t\to\infty}[T_e+(T_0-T_e)e^{-kt}] = T_e.$$

The Von Bertalanffy (Restricted) Growth Equation

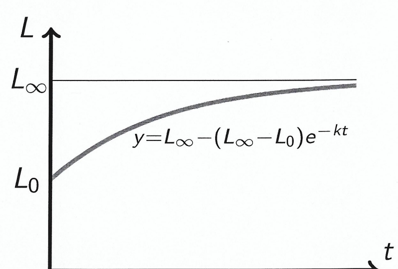
A commonly used DE for the growth, in length, of an individual fish is

$$\frac{dL}{dt} = k(L_\infty - L) \qquad L(0) = L_0,$$

where $L(t)$ is length at age t, L_∞ is the asymptotic length and k is a positive constant. The DE captures the idea that the rate of growth is proportional to the difference between asymptotic and current length.

We can show that the solution of this IVP is given by

$$\boxed{L(t) = L_\infty - (L_\infty - L_0)e^{-kt}}.$$



Notice also that

$$\lim_{t\to\infty} L(t) = \lim_{t\to\infty}[L_\infty-(L_\infty-L_0)e^{-kt}] = L_\infty.$$

Allometric Growth

In biology, **allometry** is the study of the relationship between sizes of parts of an organism (e.g., skull length and body length, or leaf area and stem diameter).

We denote by $L_1(t)$ and $L_2(t)$ the respective sizes of two organs of an individual of age t. We say that L_1 and L_2 are related through an allometric law if their specific growth rates are proportional—that is, if

$$\frac{1}{L_1}\cdot\frac{dL_1}{dt} = k\frac{1}{L_2}\cdot\frac{dL_2}{dt}$$

for some constant k. If k is equal to 1, then the growth is called isometric; otherwise it is called allometric.

We will show that he solution to this differential equation is

$$\boxed{L_1 = C\,L_2^k}$$

for some constant C.

Homeostasis

The nutrient content of a consumer can range from reflecting the nutrient content of its food to being constant. A model for homeostatic regulation is provided in Sterner and Elser (2002). It relates a consumers nutrient content (denoted by y) to its foods nutrient content (denoted by x) as

$$\frac{dy}{dx} = \frac{1}{\theta}\frac{y}{x}$$

where $\theta \geq 1$ is a constant.

We can show that $\boxed{y = C\,x^{1/\theta}}$ for some positive constant C.

Absence of homeostasis means that the consumer reflects the food's nutrient content. This occurs when $y = Cx$ and thus when $\theta = 1$.

Strict homeostasis means that the nutrient content of the consumer is independent of the nutrient content of the food; that is, $y = C$; this occurs in the limit as $\theta \longrightarrow \infty$.

Equilibria of an Autonomous DE

Many of the DEs that model biological situations are of the form

$$\frac{dy}{dx} = g(y)$$

where the right-hand side does not depend explicitly on x. (We will typically think of x as time.) The equations are called **autonomous differential equations**.

Constant solutions form a special class of solutions of autonomous differential equations. These solutions are called (point) **equilibria**.

Example For example

$$N_1(t) = 0 \qquad \text{and} \qquad N_2(t) = K$$

are constant solutions to the logistic equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right).$$

Finding Equilibria

If $\widehat{y}$ (read "y hat") satisfies

$$g(\widehat{y}) = 0$$

then $\widehat{y}$ is an equilibrium of the autonomous differential equation

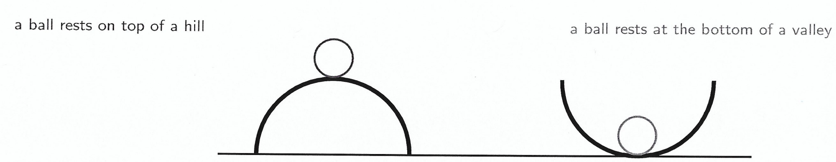
$$\frac{dy}{dx} = g(y).$$

Basic Property

The basic property of equilibria is that if, initially (say, at $x = 0$), $y(0) = \widehat{y}$ and $\widehat{y}$ is an equilibrium, then $y(x) = \widehat{y}$ for all $x \geq 0$.

Stability of Equilibria

Of great interest is the stability of equilibria of a differential equation. This is best explained by the example of a ball on a hill vs a ball in a valley:



In either case, the ball is in equilibrium because it does not move.

If we perturb the ball by a small amount (i.e., if we move it out of its equilibrium slightly) the ball on the left will roll down the hill and not return to the top, whereas the ball on the right will return to the bottom of the valley.

The ball on the **left** is **unstable** and the ball on the **right** is **stable**.

Stability for Equilibria of DE

Suppose that $\widehat{y}$ is an equilibrium of $\dfrac{dy}{dx} = g(y)$; that is, $g(\widehat{y}) = 0$.

We look at what happens to the solution when we start close to the equilibrium; that is, we consider the solution of the DE when we move away from the equilibrium by a small amount, called a *small perturbation*.

We say that $\widehat{y}$ is **locally stable** if the solution returns to the equilibrium $\widehat{y}$ after a small perturbation;

We say that $\widehat{y}$ is **unstable** if the solution does not return to the equilibrium $\widehat{y}$ after a small perturbation.

We will discuss stability of equilibria in great detail in MA 138.