MA 137 — Calculus 1 with Life Science Applications
The Definite Integral
(Section 6.1)

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Sigma (Σ) Notation

In approximating areas we have encountered sums with many terms. A convenient way of writing such sums uses the Greek letter Σ (which corresponds to our capital S) and is called *sigma notation*. More precisely, if $a_1, a_2, \ldots, a_n$ are real numbers we denote the sum

$$a_1 + a_2 + \cdots + a_n$$

by using the notation $$\sum_{k=1}^{n} a_k.$$

The integer k is called an *index* or *counter* and takes on the values $1, 2, \ldots, n$. For example,

$$\sum_{k=1}^{6} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 1 + 4 + 9 + 16 + 25 + 36 = 91$$

whereas

$$\sum_{k=3}^{6} k^2 = 3^2 + 4^2 + 5^2 + 6^2 = 9 + 16 + 25 + 36 = 86.$$

Summation Rules

The rules and formulas given next allow us to compute fairly easily Riemann sums where the number n of subintervals is rather large. We can also get compact and manageable expressions for the sum so that we can readily investigate what happens as n approaches infinity.

[sr$_1$] $$\sum_{k=1}^{n} c = n\,c$$

[sr$_2$] $$\sum_{k=1}^{n} (c\,a_k) = c \sum_{k=1}^{n} a_k$$

[sr$_3$] $$\sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$$

Note: The summations rules are nothing but the usual rules of arithmetic rewritten in the Σ notation.
For example, **[sr$_2$]** is nothing but the distributive law of arithmetic
$$c\,a_1 + c\,a_2 + \cdots + c\,a_n = c\,(a_1 + a_2 + \cdots + a_n);$$
[sr$_3$] is nothing but the commutative law of addition
$$(a_1 \pm b_1) + \cdots + (a_n \pm b_n) = (a_1 + \cdots + a_n) \pm (b_1 + \cdots + b_n).$$

Formulas [Neuhauser, Example #3 (p. 279); Problem # 31 (p. 291)]

[sf$_1$] $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

[sf$_2$] $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof: In the case of **[sf$_1$]**, let S denote the sum of the integers $1, 2, 3, \ldots, n$. Let us write this sum S twice: we first list the terms in the sum in increasing order whereas we list them in decreasing order the second time:

$$\begin{array}{ccccccccc} S & = & 1 & + & 2 & + & \cdots & + & n \\ S & = & n & + & n-1 & + & \cdots & + & 1 \end{array}$$

If we now add the terms along the vertical columns, we obtain

$$2S = \underbrace{(n+1) + (n+1) + \cdots + (n+1)}_{n \text{ times}} = n(n+1).$$

This gives our desired formula, once we divide both sides of the above equality by 2.

In the case of **[sf$_2$]**, let S denote the sum of the integers $1^2, 2^2, 3^2, \ldots, n^2$. The *trick* is to consider the sum $\sum_{k=1}^{n} [(k+1)^3 - k^3]$. On the one hand, this new sum collapses to

$$(2^3 - 1^3) + (3^3 - 2^3) + (4^3 - 3^3) + \cdots + (n^3 - (n-1)^3) + ((n+1)^3 - n^3) = (n+1)^3 - 1^3 = n^3 + 3n^2 + 3n$$

On the other hand, using our summation rules together with **[sf$_1$]** gives us

$$\sum_{k=1}^{n} [(k+1)^3 - k^3] = \sum_{k=1}^{n} [3k^2 + 3k + 1] = 3 \sum_{k=1}^{n} k^2 + 3 \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = 3S + 3\,\frac{n(n+1)}{2} + n$$

Equating the right hand sides of the above identities gives us: $3S + 3\,\dfrac{n(n+1)}{2} + n = n^3 + 3n^2 + 3n.$

If we solve for S and properly factor the terms, we obtain our desired expression.

More Formulas

The next formulas can be verified in a sequential order using the same type of trick used in the proof for **[sf$_2$]**. The proofs get increasingly more tedious.

[sf$_3$] $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

[sf$_4$] $\displaystyle\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

Example 1: (Online Homework, HW 23, # 15)

Find the numerical value of the sums below:

- $\displaystyle\sum_{j=3}^{7}(4j-1)$
- $\displaystyle\sum_{i=3}^{5}(i^2-i)$

$$\sum_{j=3}^{7}(4j-1) = [4\cdot\underline{3}-1]+[4\cdot\underline{4}-1]+[4\cdot\underline{5}-1]+[4\cdot\underline{6}-1]+[4\cdot\underline{7}-1]$$

$$= 11 + 15 + 19 + 23 + 27 = 95$$

$$\sum_{i=3}^{5}(i^2-i) = [3^2-3] + [4^2-4] + [5^2-5]$$

$$= 6 + 12 + 20 = 38$$

Example 2:

Find the numerical value of the sums below:

- $\displaystyle\sum_{j=3}^{n}(4j-1)$
- $\displaystyle\sum_{i=3}^{n}(i^2-i)$

$$\sum_{j=3}^{n}(4j-1) = \sum_{j=1}^{n}(4j-1) - \sum_{j=1}^{2}(4j-1)$$

$$= \sum_{j=1}^{n}(4j-1) - \Big[\underbrace{(4\cdot 1-1)+(4\cdot 2-1)}_{=10}\Big]$$

$$= 4\sum_{j=1}^{n} j - \sum_{j=1}^{n} 1 - 10$$

$$= 4\,\frac{n(n+1)}{2} - n - 10$$

$$= 2n(n+1) - n - 10$$

$$= 2n^2 + 2n - n - 10$$

$$= \boxed{2n^2 + n - 10}$$

$$\sum_{i=3}^{n}(i^2-i) = \sum_{i=1}^{n}(i^2-i) - \sum_{i=1}^{2}(i^2-i)$$

$$= \sum_{i=1}^{n}(i^2-i) - \Big[\underbrace{(1^2-1)+(2^2-2)}_{=0+2\,=\,2}\Big]$$

$$= \Big(\sum_{i=1}^{n} i^2\Big) - \Big(\sum_{i=1}^{n} i\Big) - 2$$

use the rules $= \dfrac{n(n+1)(2n+1)}{6} - \dfrac{n(n+1)}{2} - 2$

simplify $= \dfrac{2n^3+3n^2+n - 3n(n+1) - 12}{6}$

$$= \frac{2n^3+\cancel{3n^2}+n-\cancel{3n^2}-3n-12}{6} = \frac{2n^3-2n-12}{6}$$

$$= \boxed{\frac{n^3-n-6}{3}}$$

Back to the Area Problem: Partitions

The idea we have used so far is to "to partition" or subdivide the given interval $[a, b]$ into smaller subintervals on each of which the variable x, and thus the function $f(x)$, does not change much.

Definition of a Partition

A *partition* of an interval $[a, b]$ is a set of points $\{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}$, listed increasingly, on the x-axis with $a = x_0$ and $x_n = b$. That is:

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

These points subdivide the interval $[a, b]$ into n *subintervals*

$$[a, x_1],\ [x_1, x_2],\ [x_2, x_3],\ \ldots,\ [x_{n-1}, b].$$

The k-th subinterval is thus of the form $[x_{k-1}, x_k]$ and it has *length*

$$\Delta x_k = x_k - x_{k-1}.$$

Assumption

Set $\|P\| = \max\limits_{1\le i\le n}\{\Delta x_i\}$. We assume that our partition P is such that $\|P\| \to 0$ as $n \to \infty$. In other words, we assume that the length of the longest (and, hence, of all) subinterval(s) tend(s) to zero whenever the number of subintervals in P becomes very large.

The Definite Integral

Let $f(x)$ be a function defined on an interval $[a, b]$.

- Partition the interval $[a, b]$ in n subintervals of lengths $\Delta x_1, \ldots, \Delta x_n$, respectively.
- For $k = 1, \ldots, n$ pick a representative point c_k in the corresponding k-th subinterval.

The **definite integral of f from a to b** is defined as

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\cdot \Delta x_k = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\cdot \Delta x_k$$

and it is denoted by $\displaystyle\int_a^b f(x)\,dx.$

The sum $\sum_{k=1}^{n} f(c_k)\cdot\Delta x_k$ is called a *Riemann sum* in honor of the German mathematician Bernhard Riemann (1826-1866), who developed the above ideas in full generality. The symbol $\int$ is called the *integral sign*. It is an elongated capital S, of the kind used in the 1600s and 1700s. The letter S stands for the summation performed in computing a Riemann sum. The numbers a and b are called the *lower and upper limits of integration*, respectively. The function $f(x)$ is called the *integrand* and the symbol dx is called the *differential* of x. You can think of the dx as representing what happens to the term Δx in the limit, as the size Δx of the subintervals gets closer and closer to 0.

- The role of x in a definite integral is the one of a *dummy variable*. In fact $\int_a^b x^2\,dx$ and $\int_a^b t^2\,dt$ have the same meaning. They represent the same number.
- We recall that a limit does not necessarily exist. However:

Theorem

If f is continuous on $[a,b]$ then $\int_a^b f(x)\,dx$ exists.

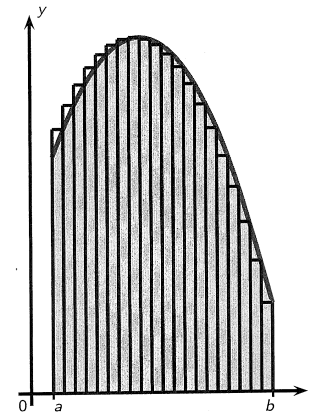
- As we observed earlier, it is computationally easier to partition the interval $[a,b]$ into n subintervals of equal length. Therefore each subinterval has length $\Delta x = \dfrac{b-a}{n}$ (we drop the index k as it is no longer necessary). In this case, there is a simple formula for the points of the partition, namely:

$$x_0 = a+0\cdot\Delta x = a,\ x_1 = a+\Delta x,\ \ldots\ x_k = a+k\cdot\Delta x,\ \ldots,\ x_n = a+n\cdot\Delta x = b$$

or, more concisely, $\quad x_k = a + k\cdot\dfrac{b-a}{n} \quad$ for $\quad k = 0,1,2,\ldots,n.$

Definite Integrals and Areas

We stress the fact that if the function f takes on only positive values then the definite integral is nothing but the area of the region below the graph of f, lying above the x-axis, and bounded by the vertical lines $x = a$ and $x = b$.

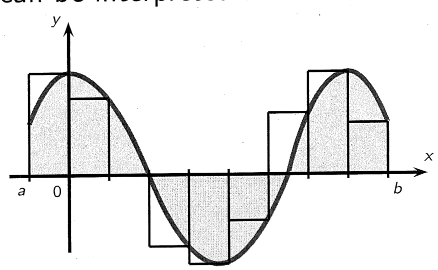


Distance traveled by an object:

If the positive valued function under consideration is the velocity $v(t)$ of an object at time t, then the area underneath the graph of the velocity function and lying above the t-axis represents the total distance traveled by the object from $t = a$ to $t = b$.

What if the Function Takes on Negative Values?

If f happens to take on both positive and negative values, then the Riemann sum is the sum of the areas of rectangles that lie above the x-axis and the negatives of the areas of rectangles that lie below the x-axis. Passing to the limit, we obtain that, in general, a definite integral can be interpreted as a difference of areas:



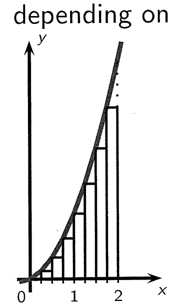
$$\int_a^b f(x)\,dx = [\text{area of the region(s) lying above the } x\text{-axis}]$$
$$-[\text{area of the region(s) lying below the } x\text{-axis}]$$

Right Versus Left Endpoint Estimates

Observe that x_k, the right endpoint of the k-th subinterval, is also the left endpoint of the $(k+1)$-th subinterval. Thus the Riemann sum estimate for the definite integral of a function f defined over an interval $[a,b]$ can be written in either of the following two forms

$$\sum_{k=0}^{n-1} f(x_k)\cdot\Delta x_{k+1} \qquad\qquad \sum_{k=1}^{n} f(x_k)\cdot\Delta x_k$$

depending on whether we use left or right endpoints, respectively.

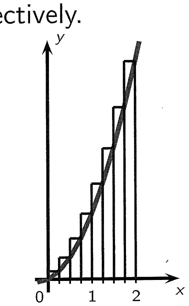


Left endpoint
Riemann sum estimate

If we are dealing with a regular partition, the above sums become

$$\sum_{k=0}^{n-1} f(a+k\cdot\Delta x)\cdot\Delta x \qquad\qquad \sum_{k=1}^{n} f(a+k\cdot\Delta x)\cdot\Delta x$$

with $\Delta x = \dfrac{b-a}{n}$ and $x_k = a + k\cdot\Delta x$ for $k = 0,1,\ldots,n.$



Right endpoint
Riemann sum estimate

Example 3: (Online Homework, HW 23, # 11)

Express the limit as a definite integral

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(5+\frac{2i}{n}\right)^{10}$$

From page 13, the formula for a Riemann sum using the right endpoints is:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f\Big(a + k\cdot \underbrace{\tfrac{b-a}{n}}_{\Delta x}\Big)\cdot \underbrace{\frac{b-a}{n}}_{\Delta x}$$

Hence in our case:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(5+\frac{2i}{n}\right)^{10}\cdot \frac{2}{n}$$

says that this is $\displaystyle\int_5^7 \underbrace{x^{10}}_{f(x)}\cdot dx$

Example 4: (Online Homework, HW 23, # 12)

Express the limit as a definite integral

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\sqrt{1+\frac{4i}{n}}$$

We can interpret $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\sqrt{1+\frac{4}{n}i}$

as $\displaystyle\int_0^4 \underbrace{\sqrt{1+x}}_{f(x)}\, dx$ (this is the type of answer that WeBWorK seeks)

(or $\displaystyle\int_1^5 \sqrt{x}\, dx$ ... which are actually equivalent)

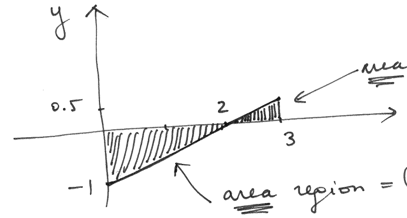
there is just an horizontal shift!

Example 5: (Online Homework, HW 23, # 7)

Evaluate the following integral by interpreting it in terms of areas:

$$\int_0^3 \left(\frac{1}{2}x - 1\right) dx$$

Let's graph the function $y = \frac{1}{2}x - 1$ on $[0,3]$



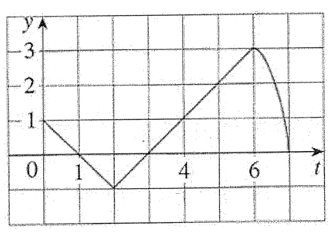
area region $= (1 \cdot \frac{1}{2}) \cdot \frac{1}{2} = \frac{1}{4}$

area region $= (2 \cdot 1) \cdot \frac{1}{2} = 1$

$$\int_0^3 \left(\frac{1}{2}x - 1\right) dx = \underbrace{\boxed{-1}}_{\text{"signed area" as it is below the x-axis}} + \frac{1}{4} = \boxed{-\frac{3}{4}}$$

Example 6: (Online Homework, HW 23, # 10)

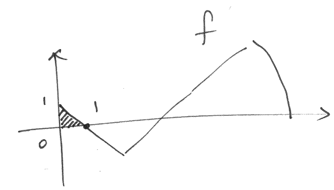
Let $g(x) = \int_0^x f(t)\, dt$, where f is the function whose graph is shown below.



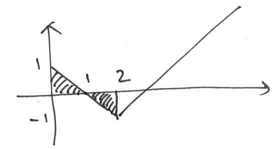
- Evaluate $g(x)$ for $x = 0, 1, 2, 3, 4, 5,$ and 6.
- Estimate $g(7)$.
- At what value of x does g attain its maximum?
- At what value of x does g attain its minimum?

$$g(x) = \int_0^x f(t)\, dt$$

$$g(0) = \int_0^0 f(t)\, dt = 0$$

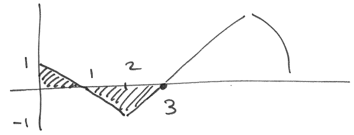


$$g(1) = \int_0^1 f(t)\, dt = \frac{1 \cdot 1}{2} = \boxed{\frac{1}{2}}$$



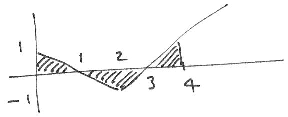
$$g(2) = \int_0^2 f(t)\, dt = \frac{1}{2} - \frac{1}{2} = \boxed{0}$$

$$g(3) = \int_0^3 f(t)\,dt$$



$$= \frac{1}{2} - 1 = \boxed{-1/2}$$

$$g(4) = \int_0^4 f(t)\,dt = \boxed{0}$$



$$= \frac{1}{2} - 1 + \frac{1}{2} = 0$$

$g(5) = \boxed{3/2}$ and $g(6) = \boxed{4}$ check.

$g(7) \cong 6.2$ (estimate)

Max of g occurs at $\boxed{x=7}$; minimum at $\boxed{x=3}$